

# Trajectory Based Models, Arbitrage and Continuity

A. Alvarez and S. E. Ferrando

Received: date / Accepted: date

**Abstract** The paper develops no arbitrage results for trajectory based models by imposing general constraints on the trading portfolios. The main condition imposed, in order to avoid arbitrage opportunities, is a local continuity requirement on the final portfolio value considered as a functional on the trajectory space. The paper shows this to be a natural requirement by proving that a large class of practical trading strategies, defined by means of trajectory based stopping times, give rise to locally continuous functionals. The theory is illustrated, with some detail, for two specific trajectory models of practical interest. The implications for stochastic models which are not semimartingales are described. The present paper extends some of the results in [1] by incorporating in the formalism a larger set of trading portfolios.

**Key words:** trajectory based arbitrage, trajectory based stopping times, local continuity, non-semimartingale models.

## 1 Introduction

There have been a few attempts to propose alternative, non probabilistic, approaches to financial market models. As examples that we are aware of, we mention [3] and [17]. The probabilistic framework requires several basic assumptions such as its reliance on sets of measure zero that, besides their technical role in the theory, have an unclear meaning in applications (to say the least). We refer to [11] for a discussion of the implications of probabilistic assumptions in finance and, in particular, the section on Knightian Uncertainty.

Reference [1] introduced a non probabilistic (NP) point of view that relies on trajectory based models and allows to prove no arbitrage results for such models. Hedging is understood as a trajectory based approximation in the case of incomplete markets; reference [1] outlines a minmax setting that allows for trajectory

---

Department of Mathematics, Ryerson University  
E-mail: alexander.alvarez@ryerson.ca  
E-mail: ferrando@ryerson.ca

based pricing in incomplete markets. A detailed analysis of this pricing methodology in the context of discrete markets is presented in [8].

We believe a trajectory based approach should illuminate and complement the stochastic approach, in particular, and following the work in [1], we provide examples of applications of our main results to non-semimartingale models. See Sections 5.3 and 6.2.

A key problem faced by a trajectory based approach is to be able to integrate with respect to functions of unbounded variation given that portfolio values, in the limit of short time intervals, are represented by such integrals. There is an array of possible integrals that can fulfil such a role (depending on the given trajectory space), we just mention [10] and [7] as examples.

The main ingredient in [1], was to place continuity restrictions on the allowed final portfolio value considered as a functional on the trajectory space. The trajectory space being treated as a topological space instead of a probability space. The present paper extends results from [1] by introducing a larger class of practical portfolios for the proposed market models. This is achieved by incorporating trajectory based stopping times which, in turn, requires the weakening of the continuity assumptions to local continuity hypothesis (we borrow the idea of local continuity from [2]). The notion of trajectory based stopping times is implicit in the usual stopping times (i.e. the formulation based on filtrations) and both notions are related ([19]); differences between the two concepts are highlighted in [4] and [12].

The paper develops results on NP-arbitrage free models and apply them to two realistic NP-market models. One of these models includes trajectories with jumps, while another one provides trajectory dependent volatility; each such trajectory class contains a large class of practical portfolios providing arbitrage free markets. In turn, each of these NP-market models can be used to obtain no arbitrage results for related non semi-martingale stochastic models.

Simple portfolios are included in the above mentioned market models; moreover, portfolios which allow for continuous re-balancing in between trajectory based stopping times are included as well. For all these portfolios, we are required to study the joint strong local continuity property of a sequence of trajectory based stopping times as a sufficient condition leading to arbitrage free NP-market models. A main technical contribution of the paper is to show that a large class of practically relevant NP-portfolios defined by means of trajectory based stopping times are locally continuous and this is achieved in different topological settings.

Fundamental results by Delbaen and Schachermayer on non-semimartingale models (see, for example, [9]) imply the existence of arbitrage strategies in the class of simple portfolios. The proposed NP-models can be applied to several non-semimartingale models indicating that local continuity is a natural sufficient condition on the allowed portfolios in order to avoid arbitrage. More details of the possible applications to non-standard stochastic frameworks can be found in [1].

The paper is organized as follows. Section 2 introduces our main definitions, in particular we provide the definition of NP-market and trajectory based stopping time and draw some basic consequences from this last concept. Section 3 introduces a notion of local continuity for a general metric space; under general assumptions, simple portfolios defined through a sequence of trajectory based stopping times are shown to define portfolios with an associated locally continuous value functional. Section 4, following [1], provides a result linking the usual (probabilistic)

notion of arbitrage with NP-arbitrage. This connection is achieved by assumptions of local continuity and small balls allowing to transfer results, back and forth, between NP-market models and stochastic market models. Section 5 constructs a NP-jump-diffusion arbitrage free market and shows how the result can be used to prove that several non-semimartingale market models are arbitrage free. Section 6 constructs a trajectory dependent volatility arbitrage free market and also draws implications to related non-semimartingale market models. Finally, Section 7 provides an overall perspective on the trajectory based approach and concludes. The Appendix contains statements and proofs of technical results used in the paper.

## 2 Non Probabilistic Framework

### 2.1 Non Probabilistic Market

Most of the definitions and ideas in this brief section were already introduced in [1], we include them here to make the paper as self-contained as possible.

Let  $x$  be a real valued function on  $[0, T]$  which is right continuous and has left limits (RCLL for short), the space of such functions will be denoted by  $\mathcal{D}[0, T]$ . We assume the existence of a non risky asset evolving with constant interest rate  $r \geq 0$  and, for simplicity, we will take  $r = 0$  for most of our arguments. The risky asset is modeled by trajectories of functions  $x$  belonging to certain class  $\mathcal{J}(x_0)$ , which we will write simply as  $\mathcal{J}$ , where  $x(0) = x_0$  for all  $x \in \mathcal{J}$ .

We also assume that, for every trajectory  $x \in \mathcal{J}(x_0) \subset \mathcal{D}[0, T]$ , the integrals

$$\int_0^t y(s, x) dx_s \quad (1)$$

are well defined for all  $t \in [0, T]$  under appropriate conditions on the integrand  $y$ . The sense in which these integrals exist is not specified yet but a general property will be assumed at this point. In the case that  $y(\cdot, x)$  is piecewise constant, namely for any  $t \in [0, T]$ :

$$y(t, x) = 1_{[0, t_1]}(t) c_0(0, x) + \sum_{i=1}^{n(x)-1} 1_{(t_i, t_{i+1}]}(t) c_i(t_i, x),$$

where  $0 = t_0 < t_1(x) < \dots < t_{n(x)}(x) = T$  is a finite,  $x$ -dependent partition, we will require that for all  $t \in [0, T]$

$$\int_0^t y(s, x) dx_s = \sum_{i=0}^{k(x)-2} c_i(t_i, x) [x_{t_{i+1}} - x_{t_i}] + c_{k-1}(t_{k-1}, x) [x_t - x_{t_{k-1}}]. \quad (2)$$

where  $k(x)$  is the smallest integer such that  $t \leq t_{k(x)}$ .

A NP-portfolio  $\Phi$  is a function  $\Phi: [0, T] \times \mathcal{J}(x_0) \rightarrow \mathbb{R}^2$ ,  $\Phi = (\psi, \phi)$ , satisfying  $\Phi(0, x) = \Phi(0, x')$  for all  $x, x' \in \mathcal{J}(x_0)$ . This common value will be denoted  $\Phi(0, x_0)$ . We will also consider the associated projections  $\Phi_x: [0, T] \rightarrow \mathbb{R}^2$  and  $\Phi_t: \mathcal{J}(x_0) \rightarrow \mathbb{R}^2$ , for fixed  $x$  and  $t$  respectively.

The value of a NP-portfolio  $\Phi$  is the function  $V_\Phi: [0, T] \times \mathcal{J}(x_0) \rightarrow \mathbb{R}$  given by:

$$V_\Phi(t, x) \equiv \psi(t, x) e^{rt} + \phi(t, x) x(t).$$

**Definition 1** Consider a class  $\mathcal{J}(x_0)$  of trajectories starting at  $x_0$  and consider a NP-portfolio  $\Phi$ :

- i)  $\Phi$  is said to be NP-predictable if  $\Phi_t(x) = \Phi_t(x')$  for all  $x, x' \in \mathcal{J}(x_0)$  such that  $x(s) = x'(s)$  for all  $0 \leq s < t$  and  $\Phi_x(\cdot)$  is left continuous and has right limits (LCRL for short) for all  $x \in \mathcal{J}(x_0)$ .
- ii)  $\Phi$  is said to be NP-self-financing if the integrals  $\int_0^t \psi(s, x) ds$  and  $\int_0^t \phi(s, x) dx_s$  exist for all  $x \in \mathcal{J}(x_0)$  in the senses of Stieltjes and expression (1) respectively and

$$V_\Phi(t, x) = V_0 + \int_0^t \psi(s, x) r ds + \int_0^t \phi(s, x) dx_s, \quad \forall x \in \mathcal{J}(x_0), \quad (3)$$

where  $V_0 = V_\Phi(0, x) = \psi(0, x) + \phi(0, x) x(0)$  for any  $x \in \mathcal{J}(x_0)$ .

*Remark 1* Consider  $r = 0$  and function  $\phi(\cdot, \cdot)$  given; define  $\Phi = (\psi, \phi)$  where  $\psi(t, x) \equiv V_\Phi(t^-, x) - x(t^-)\phi(t, x)$  and  $V_\Phi(t^-, x)$  is given by (3) with  $r = 0$ . For all the family of functions  $\phi$  considered in this paper, and under the working assumption  $r = 0$ , these portfolios  $\Phi$  will satisfy all the properties listed in Definition 1. We will not prove this fact in each instance but refer to Proposition 5 as a typical example.

**Definition 2** A NP-market model  $\mathcal{M}$  is a pair  $\mathcal{M} = (\mathcal{J}, \mathcal{A})$  where  $\mathcal{J}$  represents a class of possible trajectories for a risky asset and  $\mathcal{A}$  is a class of NP-portfolios.

For some of our results, we will need to require the following stronger hypothesis of admissibility.

**Definition 3** A NP-portfolio  $\Phi$  is said to be NP-admissible if  $V_\Phi(t, x) \geq -A$ , for a constant  $A = A(\Phi) \geq 0$ , for all  $t \in [0, T]$  and all  $x \in \mathcal{J}(x_0)$ .

The following definition provides the notion of arbitrage in a non probabilistic framework.

**Definition 4** A NP-portfolio  $\Phi$  defined on a trajectory space  $\mathcal{J}$  is a NP-arbitrage if:

- $V_0 = 0$  and  $V_\Phi(T, x) \geq 0, \forall x \in \mathcal{J}$ .
- $\exists x^* \in \mathcal{J}$  satisfying  $V_\Phi(T, x^*) > V_\Phi(0, x^*)$ .

We will say that the NP-market  $\mathcal{M} = (\mathcal{J}, \mathcal{A})$  is arbitrage free if  $\Phi$  is not a NP-arbitrage, for each  $\Phi \in \mathcal{A}$ .

## 2.2 Trajectory Based Stopping Times

The usual stopping times depend on a given filtration but a much related notion can be defined in a trajectory based sense.

**Definition 5** Let  $\mathcal{J}$  be a class of trajectories, a functional  $\tau : \mathcal{J} \rightarrow [0, T]$  is a trajectory based stopping time on  $\mathcal{J}$  if for every pair of trajectories  $x, y \in \mathcal{J}$ , with  $x(s) = y(s)$  for all  $s \in [0, \tau(x)]$ , it follows that  $\tau(y) = \tau(x)$ .

*Remark 2* Unless indicated otherwise, when proving a certain functional to be a trajectory based stopping time, its domain  $\mathcal{J}$  will be taken to be the whole set of RCLL functions  $(\mathcal{D}([0, T]))$ . This approach provides more general results because, once the result is obtained on  $\mathcal{D}([0, T])$ , it applies to any arbitrary subset.

It will take a separate study to derive systematically the consequences following from Definition 5, we content ourselves with providing some basic results, some of them will be used in the remaining of the paper. We also refer to [4] and [12] for closely related developments.

It is well known that the usual stopping times (i.e. the a filtration based formulation) can be equivalently recast in terms of trajectories. See Theorem 7 in [19] (this is sometimes referred to as Galmarino's test). As an illustration, we formulate here a particular version of this type of result. Define, for each  $s \in [0, T]$ , the functions  $X_s : \mathcal{J} \rightarrow \mathbb{R}$  by  $X_s(x) = x(s)$  and the associated canonical filtration on a given trajectory space  $\mathcal{J}$ :

$$\mathcal{F}_t^{\mathcal{J}} = \sigma(X_s : 0 \leq s \leq t),$$

where we have considered  $\mathbb{R}$  with the Borel sigma algebra. We then have the following result (see Problem 2.2 from [13]):

**Proposition 1** *A stopping time  $\tau$  relative to the filtration  $\{\mathcal{F}_t^{\mathcal{J}}\}$  is also a trajectory based stopping time.*

Proposition 1 shows that stopping times with respect to the canonical filtration associated to the process  $X = \{X_s\}_{0 \leq s \leq T}$  will also be trajectory based stopping times.

Proposition 2 provides several examples of stopping times. The notation below makes use of the convention  $\inf_{t \in [0, T]} \emptyset \equiv T$ .

**Proposition 2** *The following are trajectory based stopping times on  $\mathcal{D}([0, T])$ :*

1.  $\tau(x) = c$  if  $c \in [0, T]$ .
2.  $\tau(x) = (\tau_1(x) + \tau_2(x)) \wedge T$  where  $\tau_1, \tau_2$  are trajectory based stopping times on  $\mathcal{D}([0, T])$ .
3. Let  $A \subset \mathbb{R}$  be a closed set. For all  $x \in \mathcal{J}$ , define  $\tau(x) = \inf\{t \in [0, T] : x_t \in A\}$ . Then  $\tau$  is a trajectory based stopping time.
4.  $\tau(x) = \inf_t \{x(t) \geq a\}$ .
5. Consider  $\delta > 0$ , the following functional is a trajectory based stopping time:

$$\tau_\delta(x) = \inf_{t \in [0, T]} \{|x(t) - x(t^-)| > \delta\}.$$

6. Consider a function  $g : \mathbb{R}^N \rightarrow \mathbb{R}$  satisfying  $g(a_1, \dots, a_N) \leq a_i$ ,  $i = 1, \dots, N$  and assume that  $\tau_i$ ,  $i = 1, \dots, N$ , are trajectory based stopping times. Then,  $\tau(x) \equiv g(\tau_1(x), \dots, \tau_N(x))$  is a trajectory based stopping time. It follows then, that the minimum of a finite collection of trajectory based stopping times is also a trajectory based stopping time.
7. Let  $\tau_t$  be a collection of trajectory based stopping times indexed by  $t \in I$ , where  $I$  is an arbitrary index set. Then  $\tau(x) = \sup_{t \in I} \tau_t(x)$  is a trajectory based stopping time.

*Proof* Items 1 and 2 have immediate proofs. To prove 3, consider  $x, y \in \mathcal{D}([0, T])$  such that  $x(s) = y(s)$  for all  $s \in [0, \tau(x)]$ . In the case that  $x(\tau(x)) \in A$  we have that  $y(\tau(x)) \in A$  and  $y(s) \notin A$  for all  $s \in [0, \tau(x))$ . Hence  $\tau(y) = \tau(x)$ . The case  $x(\tau(x)) \notin A$  is not possible as we argue next; for each  $\epsilon_n > 0$  there exists  $t_n$  satisfying  $\tau(x) \leq t_n < \tau(x) + \epsilon_n$  with  $x(t_n) \in A$ . Then, this implies, by right continuity and the fact that  $A$  is closed, that  $\lim_{t_n \searrow \tau(x)} x(t_n) = x(\tau(x)) \in A$ .

The result 4 follows from 3 by taking  $A = [a, \infty)$ .

To prove 5 assume  $x, y$  to be RCLL functions satisfying  $x(s) = y(s)$  for all  $s \in [0, \tau_\delta(x)]$ . We analyze two cases: Case *i*) when  $|x(\tau_\delta(x)) - x(\tau_\delta(x)^-)| > \delta$ , it follows that  $|y(\tau_\delta(x)) - y(\tau_\delta(x)^-)| > \delta$  as well. Moreover,  $|y(t) - y(t^-)| > \delta$  for  $t \in [0, \tau(x))$  is impossible as it will contradict the definition of  $\tau_\delta(x)$  as an infimum. It follows then, that for this case,  $\tau_\delta(x) = \tau_\delta(y)$ . We will now argue that case *ii*), namely,  $|x(\tau_\delta(x)) - x(\tau_\delta(x)^-)| \leq \delta$  does not occur, this will conclude the proof. Consider  $\epsilon_n \searrow 0$ , then there exists  $\tau_\delta(x) < t_n \leq \tau_\delta(x) + \epsilon_n$  such that  $|(x(t_n) - x(t_n^-))| > \delta$ . These last statements contradict Lemma 2 (see the Appendix).

Items 6 and 7, have direct proofs.  $\square$

**Corollary 1** *Consider  $\mathcal{J}$  to be a fixed subset of  $\mathcal{D}([0, T])$  such that each  $x \in \mathcal{J}$  has a finite number of jumps. Then,*

$$\tau(x) = \inf_{t \in [0, T]} \{(x(t) - x(t^-)) \neq 0\} = \inf_{\delta > 0} \inf_{t \in [0, T]} \{|x(t) - x(t^-)| > \delta\},$$

*is a trajectory based stopping time on  $\mathcal{J}$ .*

*Proof* Clearly, from the hypothesis on finite number of jumps,  $\inf_{t \in [0, T]} \{(x(t) - x(t^-)) \neq 0\} \geq \inf_{\delta > 0} \inf_{t \in [0, T]} |x(t) - x(t^-)| > \delta$ . The reverse inequality follows by noticing that  $\inf_{t \in [0, T]} \{|x(t) - x(t^-)| > \delta\} \leq \inf_{t \in [0, T]} |x(t) - x(t^-)| > \delta$  for all  $\delta > 0$ . So,  $\tau(x) = \inf_{\delta > 0} \tau_\delta(x)$  and each  $\tau_\delta$  is a trajectory based stopping time according to Proposition 2, item 5. Again, from the hypothesis on finite number of jumps, for a fixed  $x$ , there exists  $\beta = \beta(x)$  such that  $\tau(x) = \tau_\beta(x)$ , therefore, for such fixed  $x$ , consider  $y$  such that  $x(s) = y(s)$  for all  $s \in [0, \tau(x)]$ . It follows that  $\tau_\beta(x) = \tau_\beta(y)$  and so  $\tau(x) = \tau(y)$ .  $\square$

For simplicity, and for lack of a better abbreviation, trajectory based stopping times will be referred to as NP-stopping times.

### 3 Locally Continuous Portfolios

In [2] the concept of local continuity is introduced as follows.

**Definition 6** Let  $\mathcal{X}$  and  $\mathcal{Y}$  be metric spaces. A function  $f : \mathcal{X} \rightarrow \mathcal{Y}$  is locally continuous if for all  $x \in \mathcal{X}$  there exists an open  $U_x \subset \mathcal{X}$  such that  $x \in \overline{U_x}$  and  $f(x_n) \rightarrow f(x)$  whenever  $x_n \rightarrow x$  in  $U_x$ .

The notion of local continuity is equivalent, at least in the setting of metric spaces, to quasicontinuity, a notion developed in the literature (see [14]). Local continuity is a natural topological property for NP-stopping times, this is in contrast to the stronger property of continuity. As an example we mention that the stopping time given in item 4 of Proposition 2 is not continuous with respect to the uniform metric but it is easily seen to be locally continuous relative to that metric.

In our framework, we will not only need locally continuous NP-stopping times but we will also require they satisfy the following stronger continuity property.

**Definition 7** Let  $\mathcal{J}$  be a class of trajectories provided with metric  $d$ . A stopping time  $\tau$  on  $\mathcal{J}$  is said to be strong locally continuous if for all  $x \in \mathcal{J}$  there exists an open set  $U_x \subset \mathcal{J}$  such that  $x \in \overline{U_x}$  and whenever  $x_n \rightarrow x$  in  $U_x$ :

1.  $\tau(x_n) \rightarrow \tau(x)$  (local continuity).
2.  $x_n(\tau(x_n)) \rightarrow x(\tau(x))$ .

The next proposition shows that condition 2, in Definition 7, follows from local continuity in the case of the uniform metric.

**Proposition 3** Let  $\mathcal{J}$  be a class of continuous trajectories provided with the topology induced by the uniform norm. If  $\tau$  is a locally continuous stopping time on  $\mathcal{J}$  then  $\tau$  is strong locally continuous on  $\mathcal{J}$ .

*Proof* As  $\tau$  is locally continuous on  $\mathcal{J}$ , for all  $x \in \mathcal{J}$  there exists an open  $U_x \subset \mathcal{J}$  such that  $x \in \overline{U_x}$  and  $\tau(x_n) \rightarrow \tau(x)$  whenever  $x_n \rightarrow x$  in  $U_x$ . Let  $(x_n)_{n=1,2,\dots}$  be such a sequence, then:

$$\begin{aligned} |x_n(\tau(x_n)) - x(\tau(x))| &= |x_n(\tau(x_n)) - x(\tau(x_n)) + x(\tau(x_n)) - x(\tau(x))| \\ &\leq |x_n(\tau(x_n)) - x(\tau(x_n))| + |x(\tau(x_n)) - x(\tau(x))| \\ &\leq \sup_s |x_n(s) - x(s)| + |x(\tau(x_n)) - x(\tau(x))| \\ &= \|x_n - x\| + |x(\tau(x_n)) - x(\tau(x))|. \end{aligned} \quad (4)$$

It is obvious that  $\|x_n - x\| \rightarrow 0$  as  $n \rightarrow \infty$ . On the other hand,  $\tau(x_n) \rightarrow \tau(x)$  and  $x$  is a continuous function, then  $x(\tau(x_n)) \rightarrow x(\tau(x))$ . This analysis implies that expression in (4) converges to 0, therefore  $x_n(\tau(x_n)) \rightarrow x(\tau(x))$  as  $n \rightarrow \infty$  and the strong locally continuity property is satisfied  $\square$

**Definition 8** Let  $(\mathcal{J}, \mathcal{A})$  be a NP-market. A NP-portfolio  $\Phi \in \mathcal{A}$  is said to be locally  $V$ -continuous with respect to  $d$  if the functional  $V_\Phi(T, \cdot): \mathcal{J} \rightarrow \mathbb{R}$  is locally continuous with respect to the topology induced on  $\mathcal{J}$  by the distance  $d$ .

In contrast to the use of  $V$ -continuity in [1] we will consider locally  $V$ -continuous portfolios. The reason for this is that, in general, typical NP-stopping times will generate NP-portfolios that are only locally  $V$ -continuous. The following proposition provides an example of a simple portfolio, defined by a constant NP-stopping time, that is locally  $V$ -continuous but not  $V$ -continuous. This example was considered in Propositions 4 and 5 from [1]. We briefly recall the definition of the required trajectory class  $\mathcal{J}^{a,\mu}(x_0)$  from [1].

Denote by  $\mathcal{N}([0, T])$  the collection of all functions  $n(t)$  such that there exists a non negative integer  $m$  and positive numbers  $0 < s_1 < \dots < s_m < T$  such that  $n(t) = \sum_{s_i \leq t} 1_{[0, t]}(s_i)$ . The function  $n(t)$  is considered as identically zero on  $[0, T]$  whenever  $m = 0$ .

- Given constants  $\mu, a \in \mathbb{R}$ ,  $\mu a < 0$  and  $x_0 > 0$ , let  $\mathcal{J}^{a,\mu}(x_0)$  to be the class of all functions  $x$  for which exists  $n(t) \in \mathcal{N}([0, T])$  such that:

$$x(t) = x_0 e^{\mu t} (1 + a)^{n(t)}. \quad (5)$$

The function  $n(t)$  counts the number of jumps present in the path  $x$  until, and including, time  $t$ .

**Proposition 4** Consider  $T = 1$  and the NP-portfolio with initial value  $x_0$  defined by:

- $\phi(t, x) = 1, \psi(t, x) = 0$ , for all  $x \in \mathcal{J}^{a, \mu}(x_0)$  if  $0 \leq t \leq 1/2$ .
- $\phi(t, x) = 0, \psi(t, x) = x_{\frac{1}{2}}$  for all  $x \in \mathcal{J}^{a, \mu}(x_0)$  if  $1/2 < t \leq 1$ .

Then,  $\Phi = (\psi, \phi)$  is a NP-admissible portfolio that is locally V-continuous but not continuous with respect to the Skorohod's topology on  $\mathcal{J}^{a, \mu}(x_0)$ .

*Proof* It is easy to see that  $\Phi$  is NP-admissible, the fact that it is not V-continuous is proven in Proposition 4 of [1]. Let  $x \in \mathcal{J}^{a, \mu}(x_0)$ . If  $x(1/2) - x(1/2-) = 0$  (meaning that  $x$  doesn't jump at  $t = 1/2$ ) then  $V_\Phi(T, \cdot)$  is continuous at  $x$ , hence locally continuous at  $x$ .

If  $x(1/2) - x(1/2-) \neq 0$  then consider

$$U_x^\epsilon = \{y \in \mathcal{J}^{a, \mu}(x_0) : 0 < d_S(y, x) < \epsilon, y(1/2) = x(1/2)\}.$$

Then: (i)  $x \in \overline{U}_x^\epsilon$  and (ii)  $V_\Phi(t, x_n) = V_\Phi(t, x)$  if  $x_n \rightarrow x$  in  $U_x^\epsilon$ .  $\square$

A basic class of portfolios is defined through sequences of NP-stopping times; these sequences are introduced in the following definition and the associated portfolios are introduced in Definition 11.

**Definition 9 ((Unbounded) Finite Sequence of Stopping Times)** Let  $\mathcal{J}$  be a class of trajectories and consider a non decreasing sequence  $\tau = \{\tau_n\}$  of NP-stopping times  $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$  such that, for each  $x \in \mathcal{J}$ , there is a smallest integer  $M(x)$  satisfying  $\tau_{M(x)}(x) = T$ . Such a sequence is said to be a finite sequence of stopping times.

The case of a bounded number of stopping times  $0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_N = T$  is covered by the above definition by taking  $M(x) = N$  for all  $x$ .

**Definition 10 (Joint Strong Locally Continuity)** Let  $\mathcal{J}$  be a class of trajectories provided with metric  $d$ . Consider a finite sequence of NP-stopping times  $\tau = \{\tau_n\}$  as per Definition 9. Such a sequence is said to be jointly strong locally continuous on  $\mathcal{J}$ , with respect to  $d$ , if for all  $x \in \mathcal{J}$  there exists an open set  $U_x \subset \mathcal{J}$  such that  $x \in \overline{U}_x$  and whenever  $x_n \rightarrow x$  in  $U_x$ :

- i)  $\lim_{n \rightarrow \infty} \tau_i(x_n) = \tau_i(x)$  for all  $i$ .
- ii)  $\lim_{n \rightarrow \infty} x_n(\tau_i(x_n)) = x(\tau_i(x))$  for all  $i$ .
- iii)  $\lim_{n \rightarrow \infty} M(x_n) = M(x)$ .

Definition 10 indicates that all stopping times  $\tau_i$  are strong locally continuous, not only individually, but jointly in the sense that the open subset  $U_x \subset \mathcal{J}$  is common to all stopping times.

*Remark 3* In the case of a finite number of stopping times  $M(x) = N$  for all  $x$ , and so item iii) in Definition 10 holds for any  $\mathcal{J}$  and  $d$ . Moreover, when  $N = 2$ , which represents the case of a single NP-stopping time, Definition 10 coincides with Definition 7 and so, the requirement of joint strong local continuity reduces to strong locally continuity.



**Definition 11 (Simple Portfolios)** Assume as given:  $\tau = \{\tau_n\}$  a finite sequence of stopping times as per Definition 9, functions  $\phi_0(\cdot), \phi_1(\cdot), \dots$  defined on  $\mathcal{J}$  satisfying  $\phi_i(x) = \phi_i(\hat{x})$  for any  $x, \hat{x} \in \mathcal{J}$  satisfying  $x(s) = \hat{x}(s)$ ,  $0 \leq s \leq \tau_i(x)$ , and a real number  $V_0$ . Fix  $x \in \mathcal{J}$ ,  $t \in [0, T]$  and define:

$$\phi(t, x) \equiv \phi_0(x) 1_{[0, \tau_1(x)]}(t) + \sum_{k \geq 1} \phi_k(x) 1_{(\tau_k, \tau_{k+1}]}(t). \quad (6)$$

For fixed  $x$  and  $t \in (\tau_j(x), \tau_{j+1}(x)]$ , define:

$$V(t, x) = V_0 + \sum_{k=0}^{j-1} \phi_k(x) [x(\tau_{k+1}(x)) - x(\tau_k(x))] + \phi_j(x) [x(t) - x(\tau_j(x))] \quad (7)$$

and  $V(0, x) = V_0$ . Finally, define:  $\psi(t, x) = V(t^-, x) - \phi(t, x) x(t^-)$  and NP-portfolio strategy  $\Phi = (\psi, \phi)$ .  $\Phi$  will be said to be a NP-simple portfolio associated to the sequence  $\tau = \{\tau_n\}$ .

**Proposition 5** Consider the simple portfolio  $\Phi$  associated to a sequence  $\{\tau_n\}$  as in Definition 11. Then,  $\Phi$  is a NP-portfolio that is NP-predictable and NP-self-financing.

*Proof* Notice that  $\phi(0, x) = \phi_0(0, x)$  and the dependency of  $\phi_0(0, x)$  on  $x$  is only through  $x(0)$ , it follows that  $\Phi(0, x) = \Phi(0, x')$  for any  $x, x' \in \mathcal{J}$ . To prove NP-predictability of  $\Phi$  consider  $t \in (0, T]$  and  $x(s) = y(s)$ ,  $0 \leq s < t$ ,  $x, y \in \mathcal{J}$ . Let  $n$  be the largest integer such that  $\tau_n(x) < t$ , such an integer exists because  $\tau_0(x) = 0$ ,  $\tau_{M(x)}(x) = T$  and  $\tau_i \leq \tau_{i+1}$ . It follows that  $\tau_n(x) = \tau_n(y)$ ,  $\phi(t, x) = \phi_n(x)$  and  $\tau_{n+1}(x) \geq t$ . Also  $\tau_{n+1}(y) \geq t$  otherwise  $\tau_{n+1}(y) = \tau_{n+1}(x) < t$  which contradicts our selection of  $n$ . It follows then that  $\phi(t, y) = \phi_n(y) = \phi_n(x) = \phi(t, x)$ . This reasoning also shows  $\tau_k(x) = \tau_k(y)$  and so  $x(\tau_k(x)) = y(\tau_k(y))$  for all  $0 \leq k \leq n$ ; therefore:

$$V(t, x) - V(t, y) = \phi(t, x)(x(t) - y(t)). \quad (8)$$

From the definition it follows that  $\phi(t^-, x) = \phi(t, x)$  and notice that  $V(t^-, x)$  exists because  $x(t^-)$  exists. It is also straightforward to check that  $V(t, x) - V(t^-, x) = \phi(t, x)(x(t) - x(t^-))$  which gives  $\psi(t^-, x) = \psi(t, x)$ . Using (8) we obtain  $\psi(t, x) = \psi(t, y)$ . It is also straightforward to prove that right limits exist for  $\phi$  and  $\psi$  as well. Summarizing, we have argued that  $\Phi = (\psi, \phi)$  is NP-predictable.

From the definition of  $\psi$  it follows that  $V_\Phi(t, x) = V(t, x)$  for all  $t \in [0, T]$  and all  $x \in \mathcal{J}$ . Therefore, by means means of (2), we obtain

$$V_\Phi(t, x) = V(t, x) = V_\Phi(0, x_0) + \int_0^t \phi(s, x) dx_s \text{ for all } x \in \mathcal{J},$$

where  $V_\Phi(0, x_0) = V_0$ , hence  $\Phi$  is NP-self financing.  $\square$

**Theorem 1** Let  $\tau = \{\tau_n\}$  be as in Definition 10 and consider the associated simple portfolio  $\Phi$  as in Definition 11 and further assume that the functions  $\phi_k$  appearing in (6) are continuous functions. Then,  $\Phi$  is a NP-portfolio that is NP-predictable, NP-self-financing and locally  $V$ -continuous.

*Proof* According to Proposition 5 we only need to check the locally V-continuous property. For every possible trajectory  $x \in \mathcal{J}$ , the value of the NP-portfolio  $\Phi$  at maturity time  $T$  can be expressed as:

$$V_{\Phi}(T, x) = V_{\Phi}(0, x_0) + \sum_{i=0}^{M(x)-1} \phi_i(x) [x(\tau_{i+1}(x)) - x(\tau_i(x))].$$

Consider a fixed, but arbitrary,  $x^* \in \mathcal{J}$ , as  $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$  is a jointly strong locally continuous sequence of NP-stopping times, there exists an open set  $U_{x^*}$  with  $x^* \in \overline{U_{x^*}}$  such that if  $x_n \rightarrow x^*$ , with  $x_n \in U_{x^*}$ , then  $x_n(\tau_{i+1}(x_n)) - x_n(\tau_i(x_n)) \rightarrow x^*(\tau_{i+1}(x^*)) - x^*(\tau_i(x^*))$  for all  $0 \leq i$ . Moreover, as  $M(\cdot)$  is integer valued,  $M(x_n) = M(x^*)$  for  $n$  large enough by Definition 10. Given that the functions  $\phi_0, \dots, \phi_{M(x^*)}$  are continuous there are neighborhoods  $W_i$  of  $x^*$  such that if  $x_n \rightarrow x^*$ , with  $x_n \in W_i$ , then  $\phi_i(x_n) \rightarrow \phi_i(x^*)$  for all  $0 \leq i \leq M(x^*)$ . Consider  $W = \bigcap_{i=0}^{M(x^*)} W_i$  and  $V_{x^*} \equiv W \cap U_{x^*}$ . It follows that  $x^* \in \overline{V_{x^*}}$  and hence  $V_{x^*} \neq \emptyset$ , moreover if  $x_n \rightarrow x^*$ , with  $x_n \in V_{x^*}$ , then  $V_{\Phi}(T, x_n) \rightarrow V_{\Phi}(T, x^*)$ . Therefore portfolio  $\Phi$  is locally V-continuous.  $\square$

The same proof can also be adapted to establish the following corollary.

**Corollary 2** *Consider the setting of Theorem 1 and assume  $\phi_i(x) = \hat{\phi}(x(\tau_i(x)))$  where  $\hat{\phi}: \mathbb{R}_+ \rightarrow \mathbb{R}$  is continuous. Then, the conclusions of Theorem 1 hold.*

*Remark 4* The proof of Theorem 1 can also be adapted to cover the case of  $\phi(x) = \hat{\phi}(x, \tau_i(x))$  where  $\hat{\phi}: \mathcal{J} \times [0, T] \rightarrow \mathbb{R}$  is continuous under the product topology.

#### 4 Arbitrage

This section provides a high level theorem that allows to transfer no arbitrage results from a standard, i.e. probabilistic, setting to a NP setting and vice-versa. Most of the technical details are implicit in the hypothesis which will need careful consideration in specific instances. The general approach links probabilistic and NP-portfolios therefore, at this point, we need to introduce some precisions about the hypotheses on the probabilistic models that fall under the scope of our results.

The notion of probabilistic market that we use throughout the paper is similar to the one in [2]. Assume a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  is given. Let  $Z$  be a RCLL adapted stochastic process modelling asset prices defined on this space. A portfolio strategy  $\Phi^z$  is a pair of stochastic processes  $\Phi^z = (\psi^z, \phi^z)$ . The value of a portfolio  $\Phi^z$  at time  $t$  is a random variable given by:

$$V_{\Phi^z}(t) = \psi_t^z e^{rt} + \phi_t^z Z_t.$$

A portfolio  $\Phi^z$  is self-financing if the integrals  $\int_0^t \psi_s^z(\omega) ds$  and  $\int_0^t \phi_s^z(\omega) dZ_s(\omega)$  exist  $P$ -a.s. as a Stieltjes integral and a Föllmer stochastic integral respectively and

$$V_{\Phi^z}(t) = V_{\Phi^z}(0) + r \int_0^t \psi_s^z ds + \int_0^t \phi_s^z dZ_s, \quad P - a.s.$$

From now on, and without further comments, all (stochastic) portfolios  $\Phi^z$  will be assumed to be self-financing and predictable.

**Definition 12** A portfolio  $\Phi^z$  is admissible if  $\Phi^z$  is self-financing, predictable, and there exists  $A^z = A^z(\Phi^z) \geq 0$  such that  $V_{\Phi^z}(t) \geq -A^z$   $P$ -a.s.  $\forall t \in [0, T]$ .

**Definition 13** A stochastic market defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  is a pair  $(Z, \mathcal{A}^Z)$  where  $Z$  is an adapted stochastic process modeling asset prices and  $\mathcal{A}^Z$  is a class of admissible portfolio strategies.

*Remark 5* We assume  $\mathcal{F}_0$  is the trivial sigma algebra, furthermore, without loss of generality, we will assume that the constant  $z_0 = Z(0, w)$  is fixed, i.e. we assume the same initial value for all paths. The constant  $V_{\Phi^z}(0, w)$  will also be denoted  $V_{\Phi^z}(0, z_0)$ .

The notion of arbitrage in a probabilistic market is standard (in this paper will be referred simply as *arbitrage*). Given a process  $Z$  as above, a portfolio  $\Phi^z$  is an arbitrage opportunity if:  $V_{\Phi^z}(0) = 0$  and  $V_{\Phi^z}(T) \geq 0$ ,  $P$ -a.s., and  $P(V_{\Phi^z}(T) > 0) > 0$ .  $(Z, \mathcal{A}^Z)$  is arbitrage free if  $\Phi^z$  is not an arbitrage, for all  $\Phi^z \in \mathcal{A}^Z$ .

Given a stochastic process  $Z$  and a trajectory space  $\mathcal{J}$  as above, consider the map  $Z : \Omega \rightarrow \mathbb{R}^{[0, T]}$  defined by  $Z(w)(t) = Z_t(w)$  and introduce the following two conditions:

$C_0$  :  $Z(\Omega) \subseteq \mathcal{J}$  a.s.

$C_1$  :  $Z$  satisfies a small ball property with respect to the metric  $d$  and the space  $\mathcal{J}$ , namely for all  $\epsilon > 0$  and for all  $x$  in  $\mathcal{J}$ :

$$P(d(Z, x) < \epsilon) > 0.$$

**Definition 14** Let a trajectory space  $\mathcal{J}$  and a stochastic process  $Z$  be given such that condition  $C_0$  holds. A NP-portfolio  $\Phi$  defined on  $\mathcal{J}$  and a stochastic portfolio  $\Phi^z$  are said to be isomorphic if :

$$P(\Phi^z(t, \omega) = \Phi(t, Z(\omega)) \text{ for } 0 \leq t \leq T) = 1.$$

**Theorem 2** Let a trajectory space  $\mathcal{J}$  and a stochastic process  $Z$  be given such that conditions  $C_0$  and  $C_1$  hold. Assume  $\Phi$  and  $\Phi^z$  are isomorphic, and that, furthermore,  $\Phi$  is locally  $V$ -continuous, then:

- i) If  $\Phi^z$  is not an arbitrage, then  $\Phi$  is not a NP-arbitrage portfolio.
- ii) If  $\Phi$  is not a NP-arbitrage, then  $\Phi^z$  is not an arbitrage portfolio.

*Proof* We proceed to prove i) by contradiction. Suppose then, that  $\Phi$  satisfies:  $V_\Phi(0, x_0) = 0$ ,  $V_\Phi(T, x) \geq 0$  for all  $x \in \mathcal{J}$  and there is also  $x^* \in \mathcal{J}$  such that  $V_\Phi(T, x^*) > 0$ . Therefore, given that  $\Phi^z$  is isomorphic to  $\Phi$ , there exists  $\Omega_1 \subset \Omega$  with  $P(\Omega_1) = 1$  such that  $V_{\Phi^z}(0, z_0) = V_\Phi(0, x_0) = 0$  and  $V_{\Phi^z}(T, w) = V_\Phi(T, Z(w))$  for all  $w \in \Omega_1$ . Define  $\Omega_2 = \{w \in \Omega : Z(\cdot, w) \in \mathcal{J}\}$ . Condition  $C_0$  implies that  $P(\Omega_2) = 1$ , hence  $P(\Omega_1 \cap \Omega_2) = 1$ . Consider  $w \in \Omega_1 \cap \Omega_2$ ; then it follows that  $V_{\Phi^z}(T, w) \geq 0$ , therefore  $V_{\Phi^z}(T) \geq 0$  holds  $P$ -a.s.

Consider  $f(x) \equiv V_\Phi(T, x)$  and  $\hat{x} \in V_{x^*}$ , where  $V_{x^*}$  is given as in Proposition 9 (see Appendix). Given that  $V_{x^*}$  is a nonempty open set, there exists  $\epsilon > 0$  such that  $B_\epsilon \equiv \{w : d(Z(w), \hat{x}) < \epsilon\} \subseteq V_{x^*}$  a.s. and  $P(B_\epsilon) > 0$ . Therefore  $V_{\Phi^z}(T, w) > 0$  on  $B_\epsilon \cap \Omega_1 \cap \Omega_2$  and this last set has non zero measure; this concludes the proof.

The proof of *ii*) is similarly achieved by contradiction. Assume  $\Phi^z$  is an arbitrage portfolio while  $\Phi$  is not. Notice that  $V_{\Phi^z}(0, z_0) = V_{\Phi}(0, x_0)$  and  $V_{\Phi^z}(T, w) = V_{\Phi}(T, Z(w))$  a.s. Assume now that  $V_{\Phi}(T, \hat{x}) < 0$  for some  $\hat{x} \in \mathcal{J}$ , local continuity of  $V_{\Phi}(T, \cdot)$ , an application of Proposition 9 and the small balls property gives  $V_{\Phi}(T, Z(w)) < 0$  for all  $w$  in a set of nonzero measure. This gives, as  $\Phi^z$  is isomorphic to  $\Phi$ , a contradiction, hence  $V_{\Phi}(T, x) \geq 0$  for all  $x \in \mathcal{J}$ . Moreover, using the isomorphism relationship once more, the fact that  $\Phi^z$  is an arbitrage and  $Z(\Omega) \subseteq \mathcal{J}$  a.s. it follows that there exists  $x^* \in \mathcal{J}$  such that  $V_{\Phi}(T, x^*) > 0$ . This is a contradiction and concludes our proof.  $\square$

## 5 A Non Probabilistic Jump Diffusion Class

This section defines a realistic trajectory space, denoted  $\mathcal{J}_\tau^{\sigma, C}(x_0)$ , and proves that a large collection of practical NP-portfolios, acting on  $\mathcal{J}_\tau^{\sigma, C}(x_0)$ , are no arbitrage portfolios. Implications to non semimartingale stochastic models are also developed.

Denote by  $\mathcal{Z}_\tau([0, T])$  the collection of all continuous functions  $z(t)$  such that  $[z]_t^\tau = t$  for  $0 \leq t \leq T$  and  $z(0) = 0$ . Notice that  $\mathcal{Z}_\tau([0, T])$  includes a.s. paths of Brownian motion if  $\tau$  is a refining sequence of partitions (see [15]). In the sequel we will assume that  $\tau$  is refining and that the notion of integral used in (1) is the Ito-Föllmer integral (see [10]).

Fix  $\sigma > 0$  and  $C$  a non empty set of real numbers such that  $\inf(C) > -1$ . Define  $\mathcal{J}_\tau^{\sigma, C}(x_0)$  as the class of real valued functions  $x$  on  $[0, T]$  such that there exists  $z \in \mathcal{Z}_\tau([0, T])$ ,  $n(t) \in \mathcal{N}([0, T])$  (this last class has been introduced in Section 3), and real numbers  $a_i \in C$ ,  $i = 1, 2, \dots, m$ , verifying:

$$x(t) = x_0 e^{\sigma z(t)} \prod_{i=1}^{n(t)} (1 + a_i), \quad (9)$$

where  $n(t)$  was introduced in (5).

### 5.1 Locally V-Continuous Portfolios on $\mathcal{J}_\tau^{\sigma, C}(x_0)$

The technical Lemma 1, the proof of which can be found in the Appendix, implies that for  $n$  large enough, the number of jumps in trajectory  $x_n$  between two consecutive stopping times  $\tau_i(x_n)$  and  $\tau_{i+1}(x_n)$  is the same as the number of jumps in trajectory  $x^*$  between two consecutive stopping times  $\tau_i(x^*)$  and  $\tau_{i+1}(x^*)$ . This result will be used in the following theorem.

**Theorem 3** *Let  $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$  be a jointly strong locally continuous sequence of NP-stopping times (as per Definition 10) defined on  $\mathcal{J}_\tau^{\sigma, C}(x_0)$  with respect to the Skorohod's metric. Assume that  $\inf_{c \in C} |c| > 0$ . Let  $\phi_0(\cdot, \cdot), \phi_1(\cdot, \cdot), \dots \in C^{1,1}([0, T] \times \mathbb{R})$  and consider the portfolio strategy given by  $\Phi_t = (\psi_t, \phi_t)$  where the amount invested in the stock  $\phi_t$  is such that*

$$\phi(t, x) = 1_{[\tau_0, \tau_1]}(t) \phi_0(t, x(t-)) + \sum_{i=1}^{M(x)-1} 1_{(\tau_i, \tau_{i+1}]}(t) \phi_i(t, x(t-))$$

and  $\psi_t$  is given as described in Remark 1. Then, the portfolio  $\Phi$  is NP-predictable, NP-self-financing and locally V-continuous on  $\mathcal{J}_\tau^{\sigma,C}$  relative to the Skorohod's topology.

*Proof* Using similar arguments to the ones used in Proposition 5 one can prove that a portfolio  $\Phi$  as above is NP-predictable and NP-self-financing. Next we will prove that it is also locally V-continuous.

For  $i = 0, 1, \dots$  define the functions  $U_\Phi^i : \mathbb{R}^2 \rightarrow \mathbb{R}$  as:

$$U_\Phi^i(t, y) = \int_{y_0}^y \phi_i(t, \xi) d\xi.$$

Let

$$u_\Phi(x) = \sum_{i=0}^{M(x)-1} u_\Phi^i(x),$$

where the functionals  $u_\Phi^i : \mathcal{J}_\tau^{\sigma,C} \rightarrow \mathbb{R}$  are defined as:

$$\begin{aligned} u_\Phi^i(x) &= U_\Phi^i(\tau_{i+1}(x), x(\tau_{i+1}(x))) - U_\Phi^i(\tau_i(x), x(\tau_i(x))) \\ &\quad - \int_{\tau_i(x)}^{\tau_{i+1}(x)} \frac{\partial U_\Phi^i}{\partial t}(s, x(s-)) ds - \frac{1}{2} \int_{\tau_i(x)}^{\tau_{i+1}(x)} \frac{\partial^2 U_\Phi^i}{\partial x^2}(s, x(s-)) d\langle x \rangle_s^T \\ &\quad - \sum_{\tau_i(x) < s \leq \tau_{i+1}(x)} \left[ U_\Phi^i(s, x(s)) - U_\Phi^i(s, x(s-)) - \frac{\partial U_\Phi^i}{\partial x}(s, x(s-)) \Delta x(s) \right]. \end{aligned} \quad (10)$$

The Itô-Föllmer formula from [10] allows us to obtain:

$$u_\Phi^i(x) = \int_{\tau_i(x)}^{\tau_{i+1}(x)} \frac{\partial U_\Phi^i}{\partial x}(s, x(s-)) dx(s) = \int_{\tau_i(x)}^{\tau_{i+1}(x)} \phi_i(s, x(s-)) dx(s),$$

then

$$u_\Phi(x) = \sum_{i=0}^{M(x)-1} u_\Phi^i(x) = \int_0^T \phi(s, x(s-)) dx(s).$$

For all  $x \in \mathcal{J}_\tau^{\sigma,C}$ ,  $d\langle x \rangle_s^T = \sigma^2 x^2(s-) ds$ , therefore:

$$u_\Phi^i(x) = U_\Phi^i(\tau_{i+1}(x), x(\tau_{i+1}(x))) - U_\Phi^i(\tau_i(x), x(\tau_i(x))) - I_\Phi^i(x) - S_\Phi^i(x)$$

where

$$I_\Phi^i(x) = \int_{\tau_i(x)}^{\tau_{i+1}(x)} \frac{\partial U_\Phi^i}{\partial t}(s, x(s-)) ds + \frac{1}{2} \int_{\tau_i(x)}^{\tau_{i+1}(x)} \frac{\partial^2 U_\Phi^i}{\partial x^2}(s, x(s-)) \sigma^2 x^2(s-) ds$$

and

$$S_\Phi^i(x) = \sum_{\tau_i(x) < s \leq \tau_{i+1}(x)} \left[ U_\Phi^i(s, x(s)) - U_\Phi^i(s, x(s-)) - \frac{\partial U_\Phi^i}{\partial x}(s, x(s-)) \Delta x(s) \right].$$

Fix  $x^* \in \mathcal{J}_\tau^{\sigma,C}(x_0)$ ; as  $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$  is a jointly strong locally continuous sequence of NP-stopping times, there exists an open set  $U_{x^*} \subset \mathcal{J}_\tau^{\sigma,C}(x_0)$

such that  $x^* \in \overline{U_{x^*}}$  and whenever  $x_n \rightarrow x^*$  in  $U_{x^*}$ , then, properties i), ii) and iii) from Definition 10 hold. Let  $\{x_n\}_{n \geq 1}$  be such a sequence. We have the following:

1) Using that  $U_\Phi^i$  is continuous for all  $i$ , and the jointly strong local continuity property of  $\{\tau_n\}_{n=0,1,\dots}$  we can check that

$$U_\Phi^i(\tau_{i+1}(x_n), x_n(\tau_{i+1}(x_n))) - U_\Phi^i(\tau_i(x_n), x_n(\tau_i(x_n)))$$

converges to

$$U_\Phi^i(\tau_{i+1}(x^*), x^*(\tau_{i+1}(x^*))) - U_\Phi^i(\tau_i(x^*), x^*(\tau_i(x^*)))$$

as  $n$  approaches infinity.

2) Also,  $I_\Phi^i(x_n) \rightarrow I_\Phi^i(x^*)$  as  $n$  approaches infinity. This can be proved using the same technique as in the proof of Proposition 7 in [1].

3) Along the lines of the proof of Proposition 7 in [1] we can also prove that  $S_{\Phi^i}(x_n) \rightarrow S_{\Phi^i}(x^*)$  as  $n$  approaches infinity. The only new element in the proof is the use of Lemma 1 in order to establish a correspondence between the jumps of  $x_n$  and those of  $x^*$  for  $n$  large enough.

Combining 1), 2) and 3) above we get that for all  $i$ ,  $u_\Phi^i(x_n)$  converges to  $u_\Phi^i(x^*)$ , therefore  $u_\Phi(x_n)$  converges to  $u_\Phi(x^*)$ .

This implies that

$$V_\Phi(T, x_n) = V_0 + \int_0^T \phi(s, x_n(s-)) dx_n(s) \rightarrow V_0 + \int_0^T \phi(s, x^*(s-)) dx^*(s) = V_\Phi(T, x^*)$$

therefore  $\Phi$  is locally  $V$ -continuous on  $\mathcal{J}_\tau^{\sigma, C}$  relative to the Skorohod's topology.  $\square$

The following proposition provides examples of sequences of NP-stopping times that are jointly strong locally continuous on  $\mathcal{J}_\tau^{\sigma, C}(x_0)$ .

**Proposition 6** *Let  $\{K_i\}_{i=1,2,\dots}$  be an increasing sequence of real numbers with  $K_i \rightarrow \infty$ , and  $K_i > x_0$  for all  $i$ . If  $\inf_{c \in C} |c| > 0$  then the following sequences of NP-stopping times are jointly strong locally continuous on  $\mathcal{J}_\tau^{\sigma, C}(x_0)$  with respect to the Skorohod's metrics:*

- 1)  $\tau_i(x) = \min(\frac{i}{N}T, T)$ , for  $i = 0, 1, \dots$ , and  $N \geq 1$  an arbitrary integer.
- 2)  $\tau_i(x) = \min\left(\inf_t : x_t \geq K_i, T\right)$ , for  $i = 1, 2, \dots$
- 3)  $\tau_i(x) = \min\left(\inf_t : \sum_{s \leq t} 1_{\mathbb{R} \setminus \{0\}}(x(s) - x(s-)) \geq i, T\right)$ , for  $i = 1, 2, \dots$

*Proof* See the Appendix.

## 5.2 Arbitrage-Free NP-Portfolios for Jump Diffusion Class $\mathcal{J}_\tau^{\sigma,C}$

This section proves a class of NP-portfolios to be NP-arbitrage free for the trajectory space  $\mathcal{J}_\tau^{\sigma,C}$ . Towards this end we will make use of Theorem 2 which, in turns, requires the introduction of an appropriate stochastic market model.

**Definition 15** For any  $x_0 > 0$  consider, in a probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ , an exponential jump diffusion processes, starting at  $x_0$  given by:

$$Z_t = x_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t} \prod_{i=1}^{N_t} (1 + X_i),$$

where  $W = \{W_t\}$  is a standard Brownian motion,  $N = \{N_t\}$  is a homogeneous Poisson Process with intensity  $\lambda > 0$ , and the  $X_i$  are independent random variables, also independent of  $W$  and  $N$ , with common probability distribution  $F_X$ .

Let  $\mathcal{A}_{JD}^Z$  be the class of admissible strategies (as in Definition 12) for the process  $Z$ .

The following theorem makes use of notation introduced above.

**Theorem 4** Let  $\mathcal{J}_\tau^{\sigma,C}$  be the trajectory class introduced in (9) endowed with the Skorohod's topology. Assume the random variables  $X_i$  to be integrable with common probability distribution  $F_X$  satisfying:

- 1)  $P(X_i \in C) = 1$ .
- 2) For any  $a \in C$  and for all  $\epsilon > 0$ ,  $F_X(a + \epsilon) - F_X(a - \epsilon) > 0$ .

Let  $\Phi$  denote one of the portfolios considered in Corollary 2 or Theorem 3 defined through the NP-stopping times from Proposition 6. These are NP-portfolios which we require to be NP-admissible (see Definition 3). Then, such a  $\Phi$  is not a NP-arbitrage portfolio.

*Proof* We will apply Theorem 2 to  $Z$  and  $\mathcal{J}_\tau^{\sigma,C}$ . Note that  $P(w \in \Omega : Z(w) \in \mathcal{J}_\tau^{\sigma,C}) = 1$ , this follows from our assumption 1). Therefore, hypothesis  $C_0$  in Theorem 2 is fulfilled. Our assumption 2) allows for the application of Proposition 6 in [1], therefore, we conclude that the process  $Z$  satisfies a small ball property with respect to Skorohod's metric and trajectory space  $\mathcal{J}_\tau^{\sigma,C}$ . It follows then that hypothesis  $C_1$  in Theorem 2 is fulfilled as well.

Let  $\Phi$  be one of the portfolios described in the statement of the theorem and define

$$\Phi^z(t, w) = \Phi(t, Z(w)), \quad (11)$$

notice that (11) is well defined in a set of full measure. We will argue below that  $\Phi^z \in \mathcal{A}_{JD}^Z$ ; notice that (11) shows  $\Phi$  to be isomorphic to  $\phi^z$ . Our hypothesis on the process  $Z$  allow to apply Proposition 9.9 from [6], this result establishes the existence of a probability  $\mathbb{Q}$  such that  $e^{-rt} Z_t$  is a martingale and so the probabilistic market  $(Z, \mathcal{A}_{JD}^Z)$  is arbitrage free. Therefore, elements of  $\mathcal{A}_{JD}^Z$  are not arbitrage portfolios. It then follows from Theorem 2, statement i), that  $\Phi$  is not a NP-arbitrage portfolio.

To complete the argument it remains to prove that  $\Phi^z \in \mathcal{A}_{JD}^Z$ , this is equivalent to proving that  $\Phi^z$ , as given by (11), is admissible as per Definition 12. Notice that

$\Phi^z$  is LCRL because  $\Phi$  is LCRL. Given that  $\Phi$  is a NP-portfolio, assumed to be NP-admissible, it then follows that to show admissibility of  $\Phi^z$  it is enough to show that  $\Phi_t^z$  is a predictable process. We provide the proof of this fact only for the stock component  $\phi_t^z$ ; because of the left continuity property,  $\phi_t^z$  will be predictable if it is adapted to the given filtration  $\mathcal{F} = \{\mathcal{F}_t\}$ , we prove this next. Let  $\tau$  denote one of the NP-stopping times considered in the statement of the theorem and define  $\hat{\tau}(w) = \tau(Z(w))$ , this maps is defined on a set of full measure and it is easy to show that they are stopping times with respect to  $\{\mathcal{F}_t\}$ . In particular, the simple portfolios have the form:

$$\phi^z(t, w) \equiv \phi_0^z(w) 1_{[0, \hat{\tau}_1(w)]}(t) + \sum_{k \geq 1} \hat{\phi}_k(Z_{\hat{\tau}_k(w)}(w)) 1_{(\hat{\tau}_k(w), \hat{\tau}_{k+1}(w)]}(t), \quad (12)$$

where  $\hat{\phi} : \mathbb{R}_+ \rightarrow \mathbb{R}$  is continuous and hence  $\hat{\phi}_k(Z_t(w))$  is  $\mathcal{F}_t$ -measurable. It follows that (12) is  $\mathcal{F}_t$ -measurable. A similar argument also can be used for the stochastic portfolios isomorphic to the continuing re-balancing portfolios from Theorem 3.  $\square$

A more general result can actually be proven as well.

**Corollary 3** *Assume the same hypothesis as in Theorem 4 but now consider the following class of portfolios:*

$\mathcal{A} \equiv \{\Phi : \Phi \text{ is a NP-portfolio, locally } V\text{-continuous and } \exists \Phi^z \in \mathcal{A}_{JD}^Z \text{ isomorphic to } \Phi\}.$

*Then, the NP-market  $(\mathcal{J}, \mathcal{A})$  is NP-arbitrage free.*

The proof of Corollary 3 is exactly the same as the one of Theorem 4; the point of the specialized Theorem 4 is to explicitly establish membership to  $\mathcal{A}$  for the portfolios considered in our paper. Theorem 7 from [1] provides further examples of portfolios belonging to  $\mathcal{A}$ .

### 5.3 Implications to Stochastic Frameworks

Theorem 2, item *ii*), in conjunction with Corollary 3, can be used to prove that certain stochastic models are arbitrage free. A main point to emphasize is that many of these stochastic models are not semi-martingales, moreover, the NP-portfolios defined through NP-stopping-times considered in the present paper define isomorphic stochastic portfolios in such models. Below, we provide the main steps required to obtain these type of results and refer to [1] for more details.

*Example 1 (Jump-diffusion related models)* Consider the following stochastic process, defined on a filtered space  $(\Omega, \{\mathcal{F}_t\}, P)$ ,

$$Y_t = e^{(\mu - \sigma^2/2)t + \sigma Z_t^G} \prod_{i=1}^{N_t^R} (1 + Y_i),$$

where  $Z^G$  is a continuous process satisfying  $\langle Z^G \rangle_t = t$ . Assume also that  $Z^G$  satisfies a small ball property on  $\mathcal{Z}_{\mathcal{T}}([0, T])$  with respect to the uniform norm. Examples of such processes  $Z^G$  are the processes  $Z^F$ ,  $Z^R$  and  $Z^w$  introduced in



Section 5 of [1]. The process  $N^R$  is a renewal process and the random variables  $Y_i$  are independent and also independent of  $Z^G$  and  $N^R$  with common distribution  $F_Y$ .

Consider the arbitrage free NP-market  $(\mathcal{J}_\tau^{\sigma, C}, \mathcal{A})$  introduced in Corollary 3 and define the following set of portfolios (defined on  $(\Omega, \{\mathcal{F}_t\}, P)$ ),

$$\mathcal{A}^Y \equiv \{\Phi^y : \text{admissible and } \exists \Phi \in \mathcal{A} \text{ isomorphic to } \Phi^y\}.$$

We argue next that, under appropriate conditions, the stochastic market  $(Y, \mathcal{A}^Y)$  is arbitrage free. Assume the hypothesis in Corollary 3 are satisfied hence  $(\mathcal{J}_\tau^{\sigma, C}, \mathcal{A})$  is NP-arbitrage free. Furthermore, under the assumptions:

- 1)  $P(Y_i \subset C) = 1$ ,
- 2) For any  $a \in C$  and for all  $\epsilon > 0$ ,  $F_Y(a + \epsilon) - F_Y(a - \epsilon) > 0$ .

One can use the same arguments as in the proof of Theorem 4 to show that hypothesis  $C_0$  and  $C_1$  in Theorem 2 hold; therefore, using ii) from that latter theorem one concludes that  $(Y, \mathcal{A}^Y)$  is arbitrage free.

## 6 Variable Volatility Models

Analogously to the developments in Section 5, the present section defines a class of trajectories  $J_\tau^\Sigma(x_0)$ . The set  $J_\tau^\Sigma(x_0)$  exhibits different volatilities for different trajectories, that is, the volatility curve/function is trajectory-dependent. A new metric is introduced that allows to prove that a large class of practical NP-portfolios are arbitrage free. Moreover, we draw some no-arbitrage implications for modified stochastic Heston models which include non semimartingale processes.

Let  $\Sigma \subset C[0, T]$  be a class of functions representing the possible volatility functions which we keep as general as possible for a good part of the developments. Section 6.1 requires to specify a definite class  $\Sigma$ . Also, let  $NQV[0, T] \subset C[0, T]$  as the class of all continuous functions  $d : [0, T] \rightarrow \mathbb{R}$  that have null quadratic variation on  $[0, T]$  and satisfy  $d_0 = 0$ .

Define

$$J_\tau^\Sigma(x_0) = \left\{ x \in C[0, T] : x(t) = x_0 e^{d_t + \int_0^t \sigma(s) dz(s)}, \sigma \in \Sigma, z \in \mathcal{Z}_T([0, T]), d \in NQV[0, T] \right\}$$

We assume that the integral that appears in the previous expression exists. In particular, in the examples that we discuss in this section we will assume that  $\sigma$  has finite variation, which is a sufficient condition for the existence of the integral. For a more detailed discussion on the existence of these integrals see [18].

Consider now a metric on  $J_\tau^\Sigma(x_0)$  given by

$$d_{QV}(x, y) = \|x - y\| + \left\| \frac{\partial}{\partial t} \langle x \rangle_t - \frac{\partial}{\partial t} \langle y \rangle_t \right\|,$$

where  $\|\cdot\|$  stands for the supremum norm on  $C[0, T]$ .

*Remark 6* Using Ito-Follmer's formula (see [10]) we can check that if  $x_t = x_0 e^{d_t^x + \int_0^t \sigma_x(s) dz_x(s)}$  and  $y_t = x_0 e^{d_t^y + \int_0^t \sigma_y(s) dz_y(s)}$  are two trajectories in  $J_{\mathcal{T}}^{\Sigma}(x_0)$ , then

$$d_{QV}(x, y) = \|x - y\| + \left\| x^2 \sigma_x^2 - y^2 \sigma_y^2 \right\|.$$

This means that  $x$  and  $y$  will be close in the metric  $d_{QV}$  if they are close in the uniform metric and their volatilities are also close in the uniform metric.

The following proposition gives sufficient conditions in order to establish the small balls property of some stochastic volatility models on  $J_{\mathcal{T}}^{\Sigma}(x_0)$  with respect to the metric  $d_{QV}$ .

**Proposition 7** *Let  $\Sigma \subset C[0, T]$  be a set of strictly positive functions of finite variation. Let  $Z$  be a stochastic volatility model on  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$  given by  $Z_t = x_0 e^{h_t + \int_0^t \sigma_s dW_s}$  where  $W$ ,  $h$  and  $\sigma$  are stochastic processes. The stochastic process  $h$  is also assumed to have null quadratic variation and  $h_0 = 0$ . Assume that  $P(\sigma(\omega) \in \Sigma) = 1$ , and  $\sigma$  satisfies a small balls property on  $\Sigma$  with respect to the uniform norm. Assume also that  $P(W(\omega) \in \mathcal{Z}_{\mathcal{T}}([0, T])) = 1$ , and there exists  $0 < \alpha \leq 1$  such that  $W = \alpha B + Y$  where  $B$  is a Brownian motion independent of  $Y$ ,  $\sigma$  and  $h$ . Then:*

- i)  $P\left(Z(\cdot, \omega) \in J_{\mathcal{T}}^{\Sigma}(x_0)\right) = 1$ .
- ii) For all  $y \in J_{\mathcal{T}}^{\Sigma}(x_0)$  and for all  $\epsilon > 0$ ,  $P(d_{QV}(Z(\omega), y) < \epsilon) > 0$ .

*Proof* The proof of i) is immediate from the construction of  $Z$ . To prove statement ii), notice that

$$\{\omega : d_{QV}(Z(\cdot, \omega), y) < \epsilon\} \supset A \cap B$$

where  $A$  and  $B$  are defined as

$$A = \left\{ \omega : \|Z(\cdot, \omega) - y\| < \frac{\epsilon}{2} \right\}$$

and

$$B = \left\{ \omega : \left\| \frac{\partial}{\partial t} \langle Z(\cdot, \omega) \rangle_t - \frac{\partial}{\partial t} \langle y \rangle_t \right\| < \frac{\epsilon}{2} \right\}.$$

The probability  $P(A) > 0$  as consequence of Theorem 3.1 in [16] (see also Remark 4.3 in [16]). The conditional probability  $P(B|A)$  is also positive as consequence of the small balls property of  $\sigma$  on  $\Sigma$  with respect to the uniform norm. Then, from  $P(A \cap B) = P(B|A)P(A)$ , we conclude that  $P(A \cap B) > 0$ , therefore  $P(\omega : d_{QV}(Z(\cdot, \omega), y) < \epsilon) > 0$ , for all  $y \in J_{\mathcal{T}}^{\Sigma}(x_0)$  and for all  $\epsilon > 0$ .  $\square$

*Remark 7* Similar results using a general integrator  $W$  could be obtained by assuming independence between  $\sigma$  and  $W$ , see [16] for some related results. However, from the modeling point of view, it is not desirable that  $\sigma$  and  $W$  are independent.

**Proposition 8** *Let  $\{K_i\}_{i=1,2,\dots}$  be an increasing sequence of real numbers with  $K_i \rightarrow \infty$ . The following sequences of NP-stopping times are jointly strong locally continuous in  $J_{\mathcal{T}}^{\Sigma}$  with respect to the metric  $d_{QV}$ :*

- 1)  $\tau_i(x) = \min(\frac{iT}{n}, T)$ , for  $i = 0, 1, \dots$

– 2)  $\tau_i(x) = \min \left( \inf_t : x_t \geq K_i, T \right)$ , for  $i = 1, 2, \dots$

*Proof* Fix  $x^* \in J_{\mathcal{T}}^{\Sigma}$  and define:

$$U_{x^*}^{1,\epsilon} = \left\{ y \in J_{\mathcal{T}}^{\Sigma} : 0 < d_{QV}(y, x^*) < \epsilon \right\},$$

$$U_{x^*}^{2,\epsilon} = \left\{ y \in J_{\mathcal{T}}^{\Sigma} : y(t) > x^*(t) \text{ for } t \geq \epsilon \right\}.$$

For each of the two sequences of NP-stopping times introduced above, consider  $U_{x^*}$  respectively as:

- 1)  $U_{x^*} = U_{x^*}^{1,\epsilon}$ .
- 2)  $U_{x^*} = U_{x^*}^{1,\epsilon} \cap U_{x^*}^{2,\epsilon}$ .

In both cases, a sequence  $\{x^{(n)}\}$  converging to  $x^*$  in the metric  $d_{QV}$  will be considered.

1) If the sequence  $x^{(n)} \in U_{x^*}^{1,\epsilon}$  converges to  $x^*$  in the metric  $d_{QV}$  then  $x^{(n)} \rightarrow x^*$  uniformly on  $[0, T]$ . The fact that the sequence of stopping times  $\tau_i(x) = \min(iT/n, T)$ , for  $i = 0, 1, \dots$  is strong locally continuous is an obvious consequence of the uniform convergence of  $\{x^{(n)}\}$  to  $x^*$  and the continuity of trajectory  $x^*$ .

2) Consider the sequence  $x^{(n)} \in U_{x^*}^{1,\epsilon} \cap U_{x^*}^{2,\epsilon}$  converging to  $x^*$ . Let us first prove iii) from Definition 10. Consider that  $M(x^*) = L^*$ , this is equivalent to say that  $K_{L^*} \leq \sup_{t \in [0, T]} x_t^* < K_{L^*+1}$ . As  $x^{(n)} \in U_{x^*}^{2,\epsilon}$ , it clearly follows that

$$\sup_{t \in [0, T]} x_t^{(n)} \geq \sup_{t \in [0, T]} x_t^* \geq K_{L^*}.$$

On the other hand, as  $x^{(n)}$  converges uniformly to  $x^*$ , for  $n$  large enough  $\sup_{t \in [0, T]} x_t^{(n)} < K_{L^*+1}$  too. Then we conclude that

$$K_{L^*} \leq \sup_{t \in [0, T]} x_t^{(n)} < K_{L^*+1}.$$

Therefore for  $n$  large enough  $M(x^{(n)}) = L^*$  so iii) has been proven.

Let us prove i) from Definition 10. As  $x^{(n)} \in U_{x^*}^{2,\epsilon}$  we have that  $\tau_i(x^{(n)}) \leq \tau_i(x^*)$  for all  $i$ . Now fix  $\epsilon > 0$ , then  $x^*(t) < K_i$  if  $t \leq \tau_i(x^*) - \epsilon$ . As  $x^{(n)}$  converges uniformly to  $x^*$  we also have that  $x^{(n)}(t) < K_i$  if  $t \leq \tau_i(x^*) - \epsilon$  for  $n$  large enough, which implies that  $\tau_i(x^{(n)}) > \tau_i(x^*) - \epsilon$  for  $n$  large enough. Then

$$\tau_i(x^*) - \epsilon < \tau_i(x^{(n)}) \leq \tau_i(x^*).$$

As  $\epsilon$  can be chosen as small as wanted then  $\lim_{n \rightarrow \infty} \tau_i(x^{(n)}) = \tau_i(x^*)$  for all  $i$ , thus i) is proven.

In order to prove ii) from Definition 10, notice that:

$$\left| x^{(n)}(\tau_i(x^{(n)})) - x^*(\tau_i(x^*)) \right| \leq \left| x^*(\tau_i(x^{(n)})) - x^*(\tau_i(x^*)) \right| + \left| x^{(n)}(\tau_i(x^{(n)})) - x^*(\tau_i(x^{(n)})) \right|.$$

The first term in the previous sum converges to 0 because  $x^*$  is continuous and  $\tau_i(x^{(n)}) \rightarrow \tau_i(x^*)$ . The second term also converges to 0 as consequence of the uniform convergence of  $x^{(n)}$  to  $x^*$ . Then we can conclude that  $x^{(n)}(\tau_i(x^{(n)})) \rightarrow x^*(\tau_i(x^*))$  as  $n \rightarrow \infty$ , therefore ii) is proven.  $\square$

**Theorem 5** *Let  $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$  be a jointly strong locally continuous sequence of NP-stopping times in  $J_{\mathcal{T}}^{\Sigma}(x_0)$  with respect to the metric  $d_{QV}$ . Let  $\phi_0(\cdot, \cdot), \phi_1(\cdot, \cdot), \dots \in C^{1,1}([0, T] \times \mathbb{R})$  and consider the portfolio strategy given by  $\Phi_t = (\psi_t, \phi_t)$  where the amount invested in the stock  $\phi_t$  is such that*

$$\phi(t, x) = 1_{[\tau_0, \tau_1]}(t)\phi_0(t, x(t-)) + \sum_{i=1}^{M(x)-1} 1_{(\tau_i, \tau_{i+1}]}(t)\phi_i(t, x(t-))$$

and  $\psi_t$  is given as described in Remark 1. Then, the portfolio  $\Phi$  is NP-predictable, NP-self-financing and locally V-continuous on  $J_{\mathcal{T}}^{\Sigma}(x_0)$  relative to the metric  $d_{QV}$ .

*Proof* Using similar arguments to the ones used in Proposition 5 one can prove that a portfolio  $\Phi$  as above is NP-predictable and NP-self-financing. Next we will prove that it is also locally V-continuous.

For  $i = 0, 1, \dots$  define the function  $U_{\phi}^i : \mathbb{R}^2 \rightarrow \mathbb{R}$  as:

$$U_{\phi}^i(t, y) = \int_{y_0}^y \phi_i(t, \xi) d\xi$$

Let

$$u_{\Phi}(x) = \sum_{i=0}^{M(x)-1} u_{\Phi}^i(x)$$

where the functionals  $u_{\Phi}^i : J_{\mathcal{T}}^{\Sigma}(x_0) \rightarrow \mathbb{R}$  are defined as:

$$\begin{aligned} u_{\Phi}^i(x) &= U_{\Phi}^i(\tau_{i+1}(x), x(\tau_{i+1}(x))) - U_{\Phi}^i(\tau_i(x), x(\tau_i(x))) \\ &\quad - \int_{\tau_i(x)}^{\tau_{i+1}(x)} \frac{\partial U_{\Phi}^i}{\partial t}(s, x(s-)) ds - \frac{1}{2} \int_{\tau_i(x)}^{\tau_{i+1}(x)} \frac{\partial^2 U_{\Phi}^i}{\partial x^2}(s, x(s-)) d\langle x \rangle_s^{\mathcal{T}} \end{aligned} \quad (13)$$

From Itô-Föllmer formula

$$u_{\Phi}^i(x) = \int_{\tau_i(x)}^{\tau_{i+1}(x)} \frac{\partial U_{\Phi}^i}{\partial x}(s, x(s-)) dx(s) = \int_{\tau_i(x)}^{\tau_{i+1}(x)} \phi_i(s, x(s-)) dx(s)$$

Then

$$u_{\Phi}(x) = \sum_{i=0}^{M(x)-1} u_{\Phi}^i(x) = \int_0^T \phi(s, x(s-)) dx(s) \quad (14)$$

Now fix  $x^* \in J_{\mathcal{T}}^{\Sigma}(x_0)$ . As  $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$  is a jointly strong locally continuous sequence of NP-stopping times, there exists an open  $U_{x^*} \subset J_{\mathcal{T}}^{\Sigma}(x_0)$  such that  $x^* \in \overline{U}_{x^*}$  and whenever  $x_n \rightarrow x^*$  in  $U_{x^*}$ , i), ii) and iii) of Definition 10 hold.

Using that  $U(\cdot, \cdot) \in C^{1,2}([0, T] \times \mathbb{R})$ , the continuity of  $x_n$  and  $x^*$ , and also that

$\tau_i(x_n) \rightarrow \tau_i(x^*)$  for all  $i$ , we conclude that:

$$\begin{aligned} U_{\Phi}^i(\tau_{i+1}(x_n), x_n(\tau_{i+1}(x_n))) - U_{\Phi}^i(\tau_i(x_n), x_n(\tau_i(x_n))) &\rightarrow \\ U_{\Phi}^i(\tau_{i+1}(x^*), x^*(\tau_{i+1}(x^*))) - U_{\Phi}^i(\tau_i(x^*), x^*(\tau_i(x^*))) \end{aligned} \quad (15)$$

and

$$\int_{\tau_i(x_n)}^{\tau_{i+1}(x_n)} \frac{\partial U_{\Phi}^i}{\partial t}(s, x_n(s-)) ds \rightarrow \int_{\tau_i(x^*)}^{\tau_{i+1}(x^*)} \frac{\partial U_{\Phi}^i}{\partial t}(s, x^*(s-)) ds. \quad (16)$$

On the other hand, we have that  $d\langle x_n \rangle_s^T = \frac{d\langle x_n \rangle_s^T}{ds} ds$  and  $d\langle x^* \rangle_s^T = \frac{d\langle x^* \rangle_s^T}{ds} ds$ .

The convergence of  $x_n$  to  $x^*$  in the metric  $d_{QV}$  implies that  $\frac{d\langle x_n \rangle_s^T}{ds} \rightarrow \frac{d\langle x^* \rangle_s^T}{ds}$  uniformly on  $[0, T]$ . This, together with the fact that  $U(\cdot, \cdot) \in C^{1,2}([0, T] \times \mathbb{R})$  and the convergence of  $\tau_i(x_n)$  to  $\tau_i(x^*)$  for all  $i$ , imply that

$$\int_{\tau_i(x_n)}^{\tau_{i+1}(x_n)} \frac{\partial^2 U_{\Phi}^i}{\partial x^2}(s, x_n(s-)) \frac{d\langle x_n \rangle_s^T}{ds} ds \rightarrow \int_{\tau_i(x^*)}^{\tau_{i+1}(x^*)} \frac{\partial^2 U_{\Phi}^i}{\partial x^2}(s, x^*(s-)) \frac{d\langle x^* \rangle_s^T}{ds} ds$$

or equivalently

$$\int_{\tau_i(x_n)}^{\tau_{i+1}(x_n)} \frac{\partial^2 U_{\Phi}^i}{\partial x^2}(s, x_n(s-)) d\langle x_n \rangle_s^T \rightarrow \int_{\tau_i(x^*)}^{\tau_{i+1}(x^*)} \frac{\partial^2 U_{\Phi}^i}{\partial x^2}(s, x^*(s-)) d\langle x^* \rangle_s^T \quad (17)$$

Combining expressions (15), (16) and (17) with (13) and (14) we get

$$u_{\Phi}(x_n) \rightarrow u_{\Phi}(x^*)$$

This implies that

$$V_{\Phi}(T, x_n) = V_0 + \int_0^T \phi(s, x_n(s)) dx_n(s) \rightarrow V_0 + \int_0^T \phi(s, x^*(s)) dx^*(s) = V_{\Phi}(T, x^*)$$

so  $\Phi$  is locally  $V$ -continuous on  $J_{\mathcal{T}}^{\Sigma}(x_0)$  relative to the metric  $d_{QV}$   $\square$

## 6.1 Arbitrage-Free NP-Portfolios for Heston-Type Trajectory Space $J_{\mathcal{T}}^{\Sigma}(x_0)$

This section introduces a specific volatility class  $\Sigma$  leading to an associated trajectory space  $J_{\mathcal{T}}^{\Sigma}$ ; it also describes a class of NP-portfolios that are NP-arbitrage free on this trajectory space. This is achieved by making use of Theorem 2 which, in turns, requires the introduction of an appropriate stochastic market model. This model is given by a Heston-type stochastic volatility process which is also used to define the class of volatility functions  $\Sigma$ .

$$Z_t = z_0 \exp \left( \int_0^t (\mu - \sigma_s^2/2) ds + \int_0^t \alpha \sigma_s dB_s^{(1)} + \int_0^t \sqrt{1 - \alpha^2} \sigma_s dB_s^{(2)} \right) \quad (18)$$

$$\sigma_s^2 = \bar{V}_s$$

$$\bar{V}_s = \frac{1}{h} \int_{s-h}^s V_t dt, h > 0$$

$$dV_s = k(\theta - V_s) + \xi \sqrt{\bar{V}_s} dB_s^{(2)}, V_0 = v_0,$$

where  $B^{(1)}$  and  $B^{(2)}$  are independent Brownian motions and  $0 < \alpha < 1$ . In order for  $\bar{V}_s$  to be defined when  $s < h$  we will assume that  $V_t = v_0$  for  $t \in [-h, 0]$ . To guarantee that the CIR process  $V$  remains strictly positive we will also assume that  $2k\theta \geq \xi^2$ .

The model described in (18) is very similar to the classical Heston model. The main modification is the regularization of the volatility process  $\sigma$ , which is usually defined as  $\sigma_s^2 = V_s$ . If  $h$  is small,  $V_s$  and  $\bar{V}_s$  will be close, meaning that if the Heston model fits empirical returns data, the regularized model also does. Similar arguments have been used previously in order to establish the practical validity of a model, see for example [5].

Let  $S_\sigma$  be the topological support of process  $\sigma$ , i.e. the minimal closed subset  $A$  of  $C[0, T]$  (equipped with the uniform norm topology) such that  $P(\sigma(\omega) \in A) = 1$ . Consider now the set  $\Sigma = \{x \in S_\sigma : x \text{ has finite variation}\}$ . It can be easily checked, that almost surely the trajectories of the volatility process  $\sigma$  are differentiable therefore have finite variation, which implies that  $P(\sigma(\omega) \in \Sigma) = 1$ . In particular,  $\Sigma$  is non-empty, but also that almost surely the trajectories of the price process  $Z$  belong to  $J_{\mathcal{T}}^\Sigma(z_0)$ , therefore Condition  $C_0$ , in Theorem 2, is satisfied.

That the process  $\sigma$  satisfies a small ball property on  $\Sigma$  with respect to the uniform norm is consequence of the fact that  $\Sigma$  is a subset of  $S_\sigma$ , the topological support of process  $\sigma$ . Then a direct application of Proposition 7 implies that Condition  $C_1$  is also satisfied.

By conveniently changing the drift, it can be checked that the Heston type model above is arbitrage free. Now we will transfer the no arbitrage property from this model to the NP-model  $J_{\mathcal{T}}^\Sigma(z_0)$ .

Let  $\Phi$  be a NP-admissible portfolio strategy defined on  $J_{\mathcal{T}}^\Sigma(z_0)$  that is given as described in Theorems 1 or 5. The sequence of stopping times that defines  $\Phi$  is considered as in Proposition 8. This guarantees that  $\Phi$  is locally  $V$ -continuous on  $J_{\mathcal{T}}^\Sigma(z_0)$  under the metric  $d_{QV}$ .

As the trajectories of the Heston model  $Z(\omega)$  belong a.s. to  $J_{\mathcal{T}}^\Sigma(z_0)$  then we can consider the isomorphic portfolio  $\Phi^Z$  on  $Z$ , defined a.s. by

$$\Phi^Z(t, \omega) = \Phi(t, Z(\omega)) \quad (19)$$

Portfolio  $\Phi^Z$  is admissible on  $Z$ , therefore  $\Phi^Z$  is not an arbitrage for this Heston model. Directly applying Theorem 2 we then conclude that  $\Phi$  is not a NP-arbitrage portfolio on  $J_{\mathcal{T}}^\Sigma(z_0)$ .

## 6.2 Implications to Modified Stochastic Heston Volatility Model

Let us consider a modified Heston stochastic volatility model similar to (18), the only difference is the addition of a new stochastic term  $Y_t$  as follows.

$$Z_t^m = z_0 \exp \left( \int_0^t (\mu - \sigma_s^2/2) ds + \int_0^t \alpha \sigma_s dB_s^{(1)} + \int_0^t \sqrt{1 - \alpha^2} \sigma_s dB_s^{(2)} + Y_t \right) \quad (20)$$

The stochastic process  $Y$  is assumed to be continuous, with null quadratic variation and independent of  $B^{(1)}$  and  $B^{(2)}$ . Analogously to the Heston model in (18), it can be proven that the modified process in (20) satisfies the conditions  $C_0$  and  $C_1$ , from Theorem 2, relative to the set of trajectories  $J_{\mathcal{T}}^{\Sigma}(z_0)$  and the metric  $d_{QV}$ .

We already argued for the fact that NP-portfolios  $\Phi$  as described in Theorems 1 or 5 do not constitute NP-arbitrage opportunities. The no arbitrage property will be transferred now to the modified Heston model in (20).

We consider now the isomorphic portfolio  $\Phi^Z$  defined almost surely as by (19). As  $\Phi$  is not an arbitrage on  $J_{\mathcal{T}}^{\Sigma}(z_0)$ , then Theorem 2 can be applied to conclude that  $\Phi^Z$  is not an arbitrage for the modified Heston model.

It is worth noticing that the conditions imposed on  $Y$  are not very strong, so the model becomes very flexible. For example, if  $Y = B^H$  is a fractional Brownian motion with  $1/2 < H \leq 3/4$ , the price process  $Z$  would not be a semimartingale.

## 7 Overview

The publication [1] proposes a trajectory based modeling of financial markets. The main strategy put forward in order to establish no arbitrage results is to connect the proposed trajectory based models with a classical stochastic reference market model. This connection is achieved through imposing continuity hypothesis and a density condition in the form of small balls. The present paper continues and strengthens this line of research by incorporating a richer class of practical portfolios defined through NP-stopping times. In the case of complete markets one can also establish trajectory per trajectory hedging results and define a natural minmax based pricing methodology that covers the incomplete market case as well (as described in [1]). Reference [8] contains a detailed development of this pricing technique in the discrete case for incomplete market models. Moreover, this last reference establishes a no arbitrage result, for discrete trajectory based markets, that does not require any reference to a stochastic market model.

### Appendix. Technical Results and Proofs.

**Lemma 1** *Let  $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$  be a jointly strong locally continuous sequence of NP-stopping times (as per Definition 10) defined on  $\mathcal{J}_{\tau}^{\sigma, C}(x_0)$  with respect to the Skorohod's metric. Assume that  $\inf_{c \in C} |c| > 0$ . Fix  $x^* \in \mathcal{J}_{\tau}^{\sigma, C}(x_0)$ . Then, there exists an open set  $U_{x^*} \subset \mathcal{J}_{\tau}^{\sigma, C}(x_0)$  such that  $x^* \in \overline{U_{x^*}}$  and whenever  $x_n \rightarrow x^*$  in  $U_{x^*}$  we have that:*

$$\sum_{s \in (\tau_i(x_n), \tau_{i+1}(x_n)]} 1_{\mathbb{R} \setminus \{0\}}(x_n(s) - x_n(s-))$$

converges to

$$\sum_{s \in (\tau_i(x^*), \tau_{i+1}(x^*)]} 1_{\mathbb{R} \setminus \{0\}}(x^*(s) - x^*(s-))$$

as  $n$  approaches infinity for all  $i \geq 0$ .

*Proof* Fix  $x^* \in \mathcal{J}_{\tau}^{\sigma, C}(x_0)$ . As  $0 = \tau_0 \leq \tau_1 \leq \tau_2 \leq \dots \leq T$  is jointly strong locally continuous, there exists an open set  $U_{x^*} \subset \mathcal{J}_{\tau}^{\sigma, C}(x_0)$  as in Definition 10. Let  $\{x_n\}_{n \geq 1}$  be any sequence of elements in  $U_{x^*}$  converging to  $x^*$  in the Skorohod's topology. Now we will consider two possible cases:

Case 1: Consider that  $x^*$  jumps at the points  $y_1 < y_2 < \dots < y_m$  in the open interval  $(\tau_i(x^*), \tau_{i+1}(x^*))$ . As  $x_n \rightarrow x^*$  in the Skorohod's topology, there exists an increasing function  $\lambda_n : [0, T] \rightarrow [0, T]$  with  $\lambda_n(0) = 0$  and  $\lambda_n(T) = T$  such that both  $\lambda_n(t) - t \rightarrow 0$  and  $x_n(\lambda_n(t)) - x^*(t) \rightarrow 0$  uniformly in  $[0, T]$ . We know from Lemma 2 in [1] that for  $n$  large enough the trajectory  $x_n$  jumps at the points  $\lambda_n(y_1) < \lambda_n(y_2) < \dots < \lambda_n(y_m)$ . As the sequence  $\{\tau_n\}_{n=0,1,\dots}$  is jointly strong locally continuous we have that  $\tau_i(x_n) \rightarrow \tau_i(x^*)$  and  $\tau_{i+1}(x_n) \rightarrow \tau_{i+1}(x^*)$  as  $n$  approaches infinity. On the other hand we know that  $\lambda_n(t) \rightarrow t$  for  $t \in [0, T]$  as  $n$  approaches infinity. Then we can conclude that

$$\tau_i(x_n) < \lambda_n(y_1) < \lambda_n(y_2) < \dots < \lambda_n(y_m) < \tau_{i+1}(x_n)$$

meaning that if  $n$  is large enough, for every jump of  $x^*$  in the open interval  $(\tau_i(x^*), \tau_{i+1}(x^*))$  there is a jump of  $x_n$  in  $(\tau_i(x_n), \tau_{i+1}(x_n))$ .

Case 2: Now suppose that  $x^*$  jumps exactly at the point  $\tau_{i+1}(x^*)$ . We know that for  $n$  large enough  $x_n$  will jump at the point  $\lambda_n(\tau_{i+1}(x^*))$ . The triangular inequality implies that

$$|x_n(\tau_{i+1}(x_n)) - x_n(\lambda_n(\tau_{i+1}(x^*)))| \leq |x_n(\tau_{i+1}(x_n)) - x^*(\tau_{i+1}(x^*))| + |x_n(\lambda_n(\tau_{i+1}(x^*))) - x^*(\tau_{i+1}(x^*))|$$

The first term in the right hand side converges to 0 as  $n$  approaches infinity because the sequence  $\{\tau_n\}_{n=0,1,\dots}$  is jointly strong locally continuous. The second term in the right hand side converges to 0 because  $x_n \rightarrow x^*$  in the Skorohod's topology. Then we conclude that  $|x_n(\tau_{i+1}(x_n)) - x_n(\lambda_n(\tau_{i+1}(x^*)))|$  approaches 0 as  $n$  approaches infinity, meaning that the point  $\tau_{i+1}(x_n) \geq \lambda_n(\tau_{i+1}(x^*))$ . Then we can conclude that if  $x^*$  jumps exactly at the point  $\tau_{i+1}(x^*)$ , then for  $n$  large enough, the trajectory  $x_n$  jumps at  $\lambda_n(\tau_{i+1}(x^*))$ , and this point satisfies that  $\tau_i(x_n) < \lambda_n(\tau_{i+1}(x^*)) \leq \tau_{i+1}(x_n)$ .

Cases 1 and 2 imply that

$$\sum_{s \in (\tau_i(x_n), \tau_{i+1}(x_n)]} 1_{\mathbb{R} \setminus \{0\}}(x_n(s) - x_n(s-))$$

converges to

$$\sum_{s \in (\tau_i(x^*), \tau_{i+1}(x^*)]} 1_{\mathbb{R} \setminus \{0\}}(x^*(s) - x^*(s-))$$

as  $n$  approaches infinity for all  $i \geq 0$ . □



**Lemma 2** Consider  $x$  to be in  $\mathcal{D}([0, T])$  and  $t \in [0, T]$ . Given any sequence  $t_n \searrow t$ ; then,  $\lim_{n \rightarrow \infty} (x(t_n) - x(t_n^-)) = 0$ .

*Proof* Using right continuity, given  $\epsilon > 0$  there exists  $N$  such that  $|x(t) - x(s)| \leq \epsilon/3$  for all  $s$  satisfying  $t < s \leq t_n$  and all  $n$  such that  $n \geq N$ . Then,

$$|x(t_n) - x(t_n^-)| \leq |x(t_n) - x(s)| + |x(s) - x(t_n^-)| \leq |x(t_n) - x(t)| + |x(t) - x(s)| + |x(s) - x(t_n^-)| \leq \epsilon,$$

where  $s$  satisfies  $s < t_n$  and  $|x(s) - x(t_n^-)| \leq \epsilon/3$ .  $\square$

**Proposition 9** Let  $f : \mathcal{X} \rightarrow \mathbb{R}$  be a locally continuous function. Consider  $x^* \in \mathcal{X}$  and an arbitrary open interval  $I$  such that  $f(x^*) \in I$ . Then, there exists an open set  $V_{x^*} \subset \mathcal{X}$ , with  $x^* \in \overline{V_{x^*}}$ , such that  $f(x) \in I$  for all  $x \in V_{x^*}$ .

*Proof* Consider  $x^* \in \mathcal{X}$  is such that  $f(x^*) \in I$  for a fixed open interval  $I$ . As  $f$  is locally continuous there exists an open set  $U_{x^*}$  such that  $x^* \in \overline{U_{x^*}}$ , and  $f(x_n) \rightarrow f(x^*)$  whenever  $x_n \rightarrow x^*$  with  $x_n \in U_{x^*}$ . Consider now a decreasing sequence of real positive numbers  $\epsilon_n$  with  $\epsilon_n \rightarrow 0$  as  $n \rightarrow \infty$ . Let  $B(x, \epsilon)$  stand for the ball of radius  $\epsilon$  centered at  $x$  and define the (nonempty) open sets  $U_n = U_{x^*} \cap B(x^*, \epsilon_n)$ . Suppose that, for an arbitrary  $N > 0$ , and all  $n \geq N$  there exists at least one  $x_n \in U_n$  such that  $f(x_n) \notin I$ . It follows then, that the sequence  $x_1, x_2, \dots \in U_{x^*}$ , which also converges to  $x^*$ , satisfies  $f(x_n) \not\rightarrow f(x^*) \in I$ . This last statement is a contradiction with the local continuity property of  $f$ . It follows that there is natural number  $n$  such that for all  $x \in U_n$  we have  $f(x) \in I$ . Taking  $V_{x^*} = U_n$ , and noticing  $x^* \in \overline{V_{x^*}}$ , the proposition is proven.  $\square$

#### Proof of Proposition 6.

*Proof* Fix  $x^* \in \mathcal{J}_\tau^{\sigma, C}(x_0)$ , then  $x^*(t) = x_0 e^{\sigma z^*(t)} \prod_{i=1}^{n^*(t)} (1 + a_i^*)$  for some  $z^* \in \mathcal{Z}_\tau([0, T])$ ,  $n^*(t) = \sum_i 1_{[0, t]}(s_i^*) \in \mathcal{N}([0, T])$ , and real numbers  $a_i^* \in C$ ,  $i = 1, 2, \dots, n^*(T)$ . The proof of this proposition strongly relies on finding the appropriate open sets  $U_{x^*}$  for each case. Before constructing these sets let us introduce some notation. Considering that any element  $y \in \mathcal{J}_\tau^{\sigma, C}(x_0)$  has the form  $y(t) = e^{\sigma z^y(t)} \prod_{i=1}^{n^y(t)} (1 + a_i^y)$  with  $z^y \in \mathcal{Z}_\tau([0, T])$ ,  $n^y(t) = \sum_i 1_{[0, t]}(s_i^y) \in \mathcal{N}([0, T])$ , and  $a_i^y \in C$ , define:

$$U_{x^*}^{1, \epsilon} = \left\{ y \in \mathcal{J}_\tau^{\sigma, C}(x_0) : 0 < d_S(y, x^*) < \epsilon \right\},$$

$$U_{x^*}^2 = \left\{ y \in \mathcal{J}_\tau^{\sigma, C}(x_0) : s_i^y < s_i^* \text{ for } i \leq n^*(T) \right\},$$

$$U_{x^*}^{3, \epsilon} = \left\{ y \in \mathcal{J}_\tau^{\sigma, C}(x_0) : z^*(t) - z^y(t) < 0 \text{ for } \epsilon \leq t \leq T \right\},$$

$$U_{x^*}^4 = \left\{ y \in \mathcal{J}_\tau^{\sigma, C}(x_0) : a_i^* - a_i^y < 0 \text{ for } i \leq n^*(T) \right\},$$

$$U_{x^*}^5 = \left\{ y \in \mathcal{J}_\tau^{\sigma, C}(x_0) : (s_i^y - s_i^*) a_i^* < 0 \text{ for } i \leq n^*(T) \right\}.$$

The fact that  $U_{x^*}$  for each of the three cases is an open set is a consequence of Lemma 2 in [1].

For each of the previous sequences of NP-stopping times, consider  $U_{x^*}$  respectively as:

$$1) U_{x^*} = U_{x^*}^{1,\epsilon} \cap U_{x^*}^2. \quad 2) U_{x^*} = U_{x^*}^{1,\epsilon} \cap U_{x^*}^{3,\epsilon} \cap U_{x^*}^4 \cap U_{x^*}^5. \quad 3) U_{x^*} = U_{x^*}^{1,\epsilon}.$$

In these three cases, a sequence  $\{x^{(n)}\}$  converging to  $x^*$  will be considered. As  $\{x^{(n)}\}$  converges to  $x^*$  in the Skorohod's metric, then there exists a sequence of increasing functions  $\lambda_n(t)$  satisfying  $\lambda_n(0) = 0$ ,  $\lambda_n(T) = T$  such that:

$$\left| x^*(t) - x^{(n)}(\lambda_n(t)) \right| \rightarrow 0 \quad \text{uniformly on } [0, T] \text{ as } n \rightarrow \infty \quad (21)$$

and

$$|\lambda_n(t) - t| \rightarrow 0 \quad \text{uniformly on } [0, T] \text{ as } n \rightarrow \infty \quad (22)$$

Case 1):

Consider a sequence of trajectories  $\{x^{(n)}\}$  in  $U_{x^*}^{1,\epsilon} \cap U_{x^*}^2$  converging to  $x^*$  in the Skorohod's topology.

Given that the NP-stopping times  $\tau_i(x) = \min(iT/N, T)$  do not depend on the trajectory  $x$ , properties i) and iii) in Definition 10 are clearly satisfied.

Now let us prove ii). Consider any  $s \in [0, T]$ . Then

$$\begin{aligned} \left| x^{(n)}(s) - x(s) \right| &= \left| x^{(n)}(s) - x^* \left( \lambda_n^{-1}(s) \right) + x^* \left( \lambda_n^{-1}(s) \right) - x^*(s) \right| \\ &\leq \left| x^{(n)}(s) - x^* \left( \lambda_n^{-1}(s) \right) \right| + \left| x^* \left( \lambda_n^{-1}(s) \right) - x^*(s) \right|. \end{aligned} \quad (23)$$

The term  $\left| x^{(n)}(s) - x^* \left( \lambda_n^{-1}(s) \right) \right|$  converges to 0 as  $n$  goes to infinity as consequence of (21).

From (22) we get that  $\lambda_n^{-1}(s) \rightarrow s$  as  $n$  goes to infinity, therefore if  $x^*$  is continuous at  $s$ , we obtain that  $\left| x^* \left( \lambda_n^{-1}(s) \right) - x^*(s) \right| \rightarrow 0$  as  $n$  goes to infinity.

If  $x^*$  has a jump at  $s$ , meaning that  $s = s_i^*$  for some  $i$ , we have from Lemma 2 in [1] that there exists an integer number  $N_0$  such that if  $n > N_0$ ,  $\lambda_n(s_i^*) = s_i^{x^{(n)}}$ . Moreover, as  $x^{(n)} \in U_{x^*}^2$  we have that  $s_i^{x^{(n)}} < s_i^*$ , therefore  $\lambda_n(s_i^*) < s_i^*$  and  $s_i^* < \lambda_n^{-1}(s_i^*)$ . This means that  $\lambda_n^{-1}(s_i^*)$  converges to  $s_i^*$  from the right. As  $x^*$  is right continuous and  $s = s_i^*$ , we have that  $\left| x^* \left( \lambda_n^{-1}(s) \right) - x^*(s) \right|$  also converges to 0 if  $x^*$  has a jump at  $s$ .

From (23) we conclude that  $\left| x^{(n)}(s) - x(s) \right| \rightarrow 0$  for any  $s \in [0, T]$ . In particular this will be true for  $s = \tau_i(x)$ , thus ii) is proven.

Case 2:

Consider a sequence of trajectories  $\{x^{(n)}\}$  in  $U_{x^*}^{1,\epsilon} \cap U_{x^*}^{3,\epsilon} \cap U_{x^*}^4 \cap U_{x^*}^5$  converging

to  $x^*$  in the Skorohod's topology.

We will first prove iii) in Definition 10. Consider that  $M(x^*) = L^*$ . This is equivalent to say that  $K_{L^*} \leq \sup_{t \in [0, T]} x_t^* < K_{L^*+1}$ . As  $\{x^{(n)}\} \rightarrow x^*$  in the Skorohod's

topology, it is easy to check that there exists  $N_0 \in \mathbb{N}$ , depending on  $x^*$  such that if  $n > N_0$  then  $\sup_{t \in [0, T]} x_t^{(n)} < K_{L^*+1}$ .

As trajectories  $\{x^{(l)}\}$  belong to  $U_{x^*}^{1, \epsilon}$ , from Lemma 2 in [1] we know that there exists  $N_1$  such that if  $l > N_1$ , then  $n^{(x^{(l)})}(T) = n^*(T)$ , meaning that for  $l$  large enough, trajectory  $x^{(l)}$  has exactly the same number of jumps as  $x^*$ , moreover, the jump times of  $x^{(l)}$  are close to the jump times of  $x^*$ . Additionally, as trajectories  $\{x^{(l)}\}$  belong simultaneously to  $U_{x^*}^4$  and  $U_{x^*}^5$ , it can be verified that if  $l > N_1$ , then for all  $t \in [0, T]$ :

$$\prod_{i=1}^{n^{(x^{(l)})}(t)} (1 + a_i^{x^{(l)}}) \geq \prod_{i=1}^{n^*(t)} (1 + a_i^*). \quad (24)$$

Given that  $\{x^{(l)}\}$  belongs to  $U_{x^*}^{3, \epsilon}$ , we also have that:

$$e^{\sigma z^{(x^{(l)})}(t)} > e^{\sigma z^*(t)} \quad (25)$$

for  $\epsilon \leq t \leq T$ . Combining expressions (24) and (25), we can see that if  $l > N_1$  holds, then:

$$x^{(l)}(t) = x_0 e^{\sigma z^{(x^{(l)})}(t)} \prod_{i=1}^{n^{(x^{(l)})}(t)} (1 + a_i^{x^{(l)}}) > x_0 e^{\sigma z^*(t)} \prod_{i=1}^{n^*(t)} (1 + a_i^*) = x^*(t), \quad (26)$$

for all  $t \in [\epsilon, T]$ . Therefore, for  $l$  large enough

$$\sup_{t \in [0, T]} x_t^{(l)} \geq \sup_{t \in [0, T]} x_t^* \geq K_{L^*},$$

so  $K_{L^*} \leq \sup_{t \in [0, T]} x_t^{(l)} < K_{L^*+1}$  which is equivalent to say that  $M(x^{(l)}) = L^*$  for  $l$  large enough so

$$\lim_{l \rightarrow \infty} M(x^{(l)}) = L^* = M(x^*)$$

and hence iii) in Definition 10 has been proven.

Now let us prove i) in Definition 10. From (26) we know that for  $l$  large enough it holds  $x^{(l)}(t) > x^*(t)$  for all  $t \in [\epsilon, T]$ , therefore  $\tau_i(x^{(l)}) \leq \tau_i(x^*)$ , for  $i = 1, 2, \dots, M(x^*)$ . Fix now  $\epsilon > 0$ , for any  $t$  such that  $\tau_i(x^*) - \epsilon < t < \tau_i(x^*)$ , the definition of  $\tau_i$  implies that  $x^*(s) < K_i$  for all  $0 \leq s \leq t$ . Then, the convergence of  $x^{(l)}$  to  $x^*$  in the Skorohod's metric implies that for  $l$  large enough,  $x^{(l)}(s) < K_i$  for all  $0 \leq s \leq \lambda_l(t)$ , meaning that for  $l$  large enough  $\lambda_l(t) < \tau_i(x^{(l)})$ . On the other hand, as  $\lambda_l$  is strictly increasing, we have  $\lambda_l(\tau_i(x^*) - \epsilon) < \lambda_l(t)$ . All this implies that for  $l$  large enough

$$\lambda_l(\tau_i(x^*) - \epsilon) < \lambda_l(t) < \tau_i(x^{(l)}) \leq \tau_i(x^*) \quad (27)$$

therefore

$$\lambda_l(\tau_i(x^*) - \epsilon) - \tau_i(x^*) < \tau_i(x^{(l)}) - \tau_i(x^*) \leq 0.$$

When  $l$  approaches infinity the expression in the left hand side approaches  $-\epsilon$ . As  $\epsilon$  can be chosen as small as we want, then the Squeeze Theorem implies that  $\tau_i(x^{(l)}) \rightarrow \tau_i(x^*)$  as  $l$  approaches infinity, thus i) is proven.

In order to prove ii) in Definition 10, notice that the triangle inequality gives

$$\begin{aligned} \left| x^*(\tau_i(x^*)) - x^{(l)}(\tau_i(x^{(l)})) \right| &\leq \left| x^*(\tau_i(x^*)) - x^*(\lambda_l^{-1}(\tau_i(x^{(l)}))) \right| \\ &\quad + \left| x^*(\lambda_l^{-1}(\tau_i(x^{(l)}))) - x^{(l)}(\tau_i(x^{(l)})) \right| \end{aligned} \quad (28)$$

As a consequence of (27) we obtain

$$\tau_i(x^*) - \epsilon < \lambda_l^{-1}(\tau_i(x^{(l)}))$$

As  $\epsilon$  can be chosen as small as wanted, it follows that  $\lambda_l^{-1}(\tau_i(x^{(l)}))$  approaches  $\tau_i(x^*)$  from the right as  $l$  approaches infinity. Then, the right continuity of  $x^*$  implies that

$$\left| x^*(\tau_i(x^*)) - x^*(\lambda_l^{-1}(\tau_i(x^{(l)}))) \right| \rightarrow 0 \text{ as } l \rightarrow \infty$$

On the other hand

$$\left| x^*(\lambda_l^{-1}(\tau_i(x^{(l)}))) - x^{(l)}(\tau_i(x^{(l)})) \right| \rightarrow 0 \text{ as } l \rightarrow \infty$$

as a consequence of the convergence of  $x^{(l)}$  to  $x^*$  in the Skorohod's metric.

As both terms in the right hand side of (28) converge to 0, then the left hand side also converges to 0, so ii) is proven.

Case 3):

From Lemma 2 in [1] it follows that for any sequence  $x^{(n)}$  converging to  $x^*$ :

- i)  $\lim_{n \rightarrow \infty} M(x^{(n)}) = M(x^*)$ .
- ii)  $\lim_{n \rightarrow \infty} \tau_i(x^{(n)}) = \tau_i(x^*)$ .

Again, as a consequence of Lemma 2 in [1], there exists an integer number  $N_0$  such that if  $n > N_0$ ,  $\lambda_n(s_i^*) = s_i^{x^{(n)}}$ . Therefore, if  $n > N_0$  we have:

$$\left| x^*(s_i^*) - x^{(n)}(\lambda_n(s_i^*)) \right| = \left| x^*(s_i^*) - x^{(n)}(s_i^{x^{(n)}}) \right|$$

Using (21) we have that  $\left| x^*(s_i^*) - x^{(n)}(s_i^{x^{(n)}}) \right| \rightarrow 0$ , so:

- iii)  $\lim_{n \rightarrow \infty} x_n(\tau_i(x_n)) = x^*(\tau_i(x^*))$ .

Therefore, the joint strong local continuity property has been proven.  $\square$

## References

1. A. Alvarez, S. Ferrando and P. Olivares, Arbitrage and Hedging in a non probabilistic framework, *Mathematics and Financial Economics*, 2012
2. C. Bender, T. Sottinen and E. Valkeila, Pricing by hedging and no-arbitrage beyond semimartingales *Finance and Stochastics* **12**, 441–468, 2008.
3. A. Bick and W. Willinger, Dynamic spanning without probabilities, *Stochastic processes and their Applications* **50** 349–374, 1994.
4. F.A. Boshizen and T.P. Hill, Moment-based minimax stopping functions for sequences of random variables, *Stochastic Processes and Applications*, 43, 303–316, 1992.
5. P. Cheridito, Mixed fractional Brownian motion, *Bernoulli*, 7(6), p. 913–934, 2001.
6. R. Cont and P. Tankov, Financial Modelling with Jump Processes, Chapman & Hall, CRC, 2004.
7. R. Cont and D. Fournie, Change of variable formulas for non-anticipative functionals on path space. <http://arxiv.org/abs/1004.1380v1>, 2010.
8. I. Degano, S. Ferrando, A. Gonzalez and M. Rahsepar, Discrete, Non Probabilistic Market Models. Arbitrage and Pricing Intervals, *in preparation*, 2013.
9. F. Delbaen and W. Schachermayer, A general version of the fundamental theorem of asset pricing, *Math. Ann.*, **300**, 463–520, 1994.
10. H. Föllmer, Calcul d'Itô sans probabilité. Seminaire de Probabilité XV. Lecture Notes in Math. No. 850. Springer Berlin, 143–150, 1981.
11. H. Föllmer and A. Schied, Probabilistic aspects of finance. Available at SSRN: <http://ssrn.com/abstract=2144072> or <http://dx.doi.org/10.2139/ssrn.2144072>, September 2012.
12. T.P. Hill and V.C. Pestin, The advantage of using non-measurable stop rules, *Ann. Probab.* 11, 442–450, 1983.
13. I. Karatzas and W. Shreve, Brownian Motion and Stochastic Calculus. Second Edition. Springer Verlag. Graduate texts in Mathematics, 113, 1998.
14. S. Kempisty, Sur les fonctions quasicontinues, *Fund Math* **19**, 184–187, 1932.
15. R. Klein and E. Giné, On quadratic variation of processes with Gaussian increments, *The Annals of Probability* **3**, N. 4, 716–721, 1975.
16. M. Pakkanen, Stochastic integrals and conditional full support, *Journal of Applied Probability* 47(3), 650–667, 2010.
17. F. Riedel, Finance without probabilistic prior assumptions. <http://arxiv.org/abs/1107.1078>, July 2011.
18. A. Schied, Model-free CPPI, to appear in *Journal of Economic Dynamics and Control*, 2013
19. A.N. Shiryaev, Optimal Stopping Rules. Springer Verlag. Stochastic Modelling and Applied Probability, 8, 2007.